TMA4295 Statistical inference
Exercise 2 - solution

Problem 1
$X \sim \Gamma(\alpha, \beta)$

a) To find the distribution of cX we can use theorem 2.1.5.

$$y = cx \Rightarrow x = \frac{y}{c} \Rightarrow \frac{dx}{dy} = \frac{1}{c}$$

$$\Rightarrow f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{y^{\alpha-1}}{c^{\alpha-1}} e^{-\frac{y}{c\beta}} \frac{1}{c} = \frac{1}{\Gamma(\alpha)(c\beta)^\alpha} y^{\alpha-1} e^{-\frac{y}{c\beta}} \sim \Gamma(\alpha, c\beta)$$

$E(Y) = cE(Y) = c\alpha\beta$
$Var(Y) = c^2 Var(X) = c^2\alpha\beta^2$.

b) $\alpha = \frac{p}{2}$ and $c = \frac{2}{\beta}$

$$\Rightarrow f_Y(y) = \frac{1}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{2^{p/2}} y^{p/2-1} e^{-\frac{y}{2}} \sim \chi^2_p$$

since Y it also a $\Gamma(\frac{p}{2}, 2)$ then from part (a) we have that $bY \sim \Gamma\left(\frac{p}{2}, 2b\right)$

c) $Z_1, Z_2, ..., Z_n \sim N(\mu, \sigma^2)$ and S^2 variance estimator,
Recall that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

and from the from the previous point we also know that $\frac{(n-1)S^2}{\sigma^2} \sim \Gamma\left(\frac{n-1}{2}, 2b\right)$

$$\Rightarrow S^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$$

$Var(S^2) = \frac{n-1}{2} \frac{4\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}$

d) $X \sim \Gamma(\alpha, \beta)$ and $k > -\alpha$

$$E(X^k) = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+k-1} e^{-\frac{x}{\beta}} dx = \frac{\beta^k}{\Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha+k-1}}{\beta^{\alpha+k}} e^{-\frac{x}{\beta}} dx = \frac{\beta^k}{\Gamma(\alpha)} \Gamma(\alpha + k).$$

To compute $E(S)$ we can use the previous part with $k = 1/2$ since S^2 has a gamma distribution. Hence

$$E(S) = \frac{\Gamma(\frac{n-1}{2} + \frac{1}{2}) \left(\frac{2\sigma^2}{n-1}\right)^{1/2}}{\Gamma(\frac{n-1}{2})} = \frac{\Gamma(\frac{n}{2}) 2^{1/2} \sigma}{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}$$

e) We can then choose as unbiased estimator

$$\hat{S} = \frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2}) 2^{1/2}} S.$$

$$Var(\hat{S}) = \left(\frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2}) 2^{1/2}}\right)^2 Var(S) = \left(\frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2}) 2^{1/2}}\right)^2 (E(S^2) - E(S)^2) =$$

$$= \left(\frac{\Gamma(\frac{n-1}{2})(n-1)^{1/2}}{\Gamma(\frac{n}{2}) 2^{1/2}}\right)^2 (\sigma^2 - (\sigma/c)^2) = \sigma^2 \left(\frac{\Gamma(\frac{n-1}{2})^2 (n-1)}{\Gamma(\frac{n}{2}) 2} - 1\right)$$

Problem 3.30

a) Let's rewrite the pdf of a binomial random variable

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} (1-p)^n e^{x \log(\frac{p}{1-p})}$$

that is an exponential family with

$$c(p) = (1-p)^n, \quad t(x) = x, \quad \omega(p) = \log(\frac{p}{1-p}), \quad h(x) = \binom{n}{x}$$

$$\omega'(p) = \frac{1}{p(1-p)} \qquad \omega''(p) = \frac{2p-1}{p^2(1-p)^2}$$

$$\frac{d}{dp} \log c(p) = -\frac{n}{1-p} \qquad \frac{d^2}{dp^2} \log c(p) = -\frac{n}{(1-p)^2}$$

Theorem 3.4.2 implies that

$$Var\left(\frac{1}{p(1-p)} X\right) = \frac{n}{(1-p)^2} - E\left(\frac{X(2p-1)}{p^2(1-p)^2}\right)$$

$$\Rightarrow \frac{1}{p^2(1-p)^2} Var(X) = \frac{n}{(1-p)^2} - \frac{np(2p-1)}{p^2(1-p)^2}$$

$$\Rightarrow Var(x) = np^2 - 2np^2 + np = np(1-p).$$

3.28

Exponential family: $f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})e^{\sum\limits_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x))}$

a) μ known: $h(x) = 1$, $c(\sigma^2) = \frac{1}{\sqrt{2\pi\sigma}}$, $w_1(\sigma^2) = -\frac{1}{2\sigma^2}$, $t_1(x) = (x-\mu)^2$

σ^2 known: $h(x) = e^{-\frac{(x)^2}{2\sigma^2}}$, $c(\mu) = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(\mu)^2}{2\sigma^2}}$, $w_1(\mu) = \mu$, $t_1(x) = \frac{x}{\sigma^2}$

b) α known: $h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$, $c(\beta) = \frac{1}{\beta^\alpha}$, $w_1(\beta) = \frac{1}{\beta}$, $t_1(x) = -x$

β known: $h(x) = e^{-\frac{x}{\beta}}$, $c(\alpha) = \frac{1}{\Gamma\alpha\beta^\alpha}$, $w_1(\alpha) = \alpha - 1$, $t_1(x) = \log(x)$

α, β unknown: $h(x) = 1$, $c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$, $w_1(\alpha) = \alpha - 1$, $w_2(\beta) = -\frac{1}{\beta}$, $t_1(x) = \log(x)$, $t_2(x) = x$

d) $h(x) = \frac{1}{x!}$, $c(\theta) = e^{-\theta}$, $w_1(\theta) = \log(\theta)$, $t_1(x) = x$

3.39
The exercise can be solved for $\mu = 0$ and $\sigma^2 = 1$ and using the substitution $z = \frac{x-\mu}{\sigma}$ afterwards, since we are working with the location-scale family.

a) Since the pdf is symmetrical around 0, 0 must be median. Verifying this, write

$$\mathrm{P}\left(Z \geq 0\right) = \int_0^\infty \frac{1}{\pi}\frac{1}{1+z^2} dz = \frac{1}{\pi}\tan^{-1}(z)\Big|_0^\infty = \frac{1}{2}$$

b) $\mathrm{P}\left(Z \geq 1\right) = \frac{1}{4}$ which also holds for $\mathrm{P}\left(Z \leq -1\right)$ by symmetry.