TMA4295 Statistical inference
Exercise 3 - solution

Problem 3.46
We first look at $x \sim unif(0,1)$, and so $E(x) = \mu_x = \frac{1}{2}$ and $Var(x) = \sigma_x^2 = \frac{1}{12}$.

$$P(|x-\mu_x| \geq k\sigma_x) = 1 - P(|x-\mu_x| < k\sigma_x) = 1 - P(\mu_x - k\sigma_x < x < \mu_x + k\sigma_x)$$

$$= \begin{cases} 1 - \frac{2k}{\sqrt{12}} & k < \sqrt{3} \\ 0 & \text{otherwise} \end{cases}$$

Now we look at $x \sim exponential(\lambda)$ and so $E(x) = \mu_x = \frac{1}{\lambda}$ and $Var(x) = \sigma_x^2 = \frac{1}{\lambda^2}$.

$$P(|x-\mu_x| \geq k\sigma_x) = P(x \geq \mu_x + k\sigma_x) + P(x \leq \mu_x - k\sigma_x) = P\left(x \geq \frac{1}{\lambda} + \frac{k}{\lambda}\right) + P\left(x \leq \frac{1}{\lambda} - \frac{k}{\lambda}\right)$$

$$= \int_{\frac{1}{\lambda}+\frac{k}{\lambda}}^{\infty} \lambda e^{-\lambda x} dx + \int_0^{\frac{1}{\lambda}-\frac{k}{\lambda}} \lambda e^{-\lambda x} dx = e^{-(1+k)} + \begin{cases} 1 - e^{k-1} & k < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{-(1+k)} + 1 - e^{k-1} & k < 1 \\ e^{-(1+k)} & k > 1 \end{cases}$$

Using Chebychev's inequality we would get:

$$P(|x-\mu_x| \geq k\sigma_x) \leq \frac{1}{k^2}.$$

Problem 3.47
We need to prove that $P(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{t+1} e^{t^2/2}$ so we first observe that

$$P(|Z| \geq t) = 2P(Z \geq t) = 2\frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2} dx = \sqrt{\frac{2}{\pi}} \int_t^{\infty} \frac{1+x^2}{1+x^2} e^{-x^2/2} dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_t^{\infty} \frac{1}{1+x^2} e^{-x^2/2} dx + \int_t^{\infty} \frac{x^2}{1+x^2} e^{-x^2/2} dx \right].$$

Now we solve by parts $\int_t^{\infty} \frac{x^2}{1+x^2} e^{-x^2/2} dx$

$$\int_t^{\infty} \frac{x^2}{1+x^2} e^{-x^2/2} dx = \frac{x}{1+x^2} \left(-e^{-x^2/2}\right)\Big|_t^{\infty} + \int_t^{\infty} \frac{1-x^2}{(1+x^2)^2} e^{-x^2/2} dx = \frac{t}{1+t^2} e^{-t^2/2} + \int_t^{\infty} \frac{1-x^2}{(1+x^2)^2} e^{-x^2/2} dx$$

Hence

$$P(|Z| \geq t) = \sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} e^{-t^2/2} + \sqrt{\frac{2}{\pi}} \int_t^{\infty} \left(\frac{1}{1+x^2} + \frac{1-x^2}{(1+x^2)^2} \right) e^{-x^2/2} dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} e^{-t^2/2} + \sqrt{\frac{2}{\pi}} \int_t^{\infty} \left(\frac{2}{(1+x^2)^2} \right) e^{-x^2/2} dx$$

$$\geq \sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} e^{-t^2/2}.$$

4.1
Since the distribution is uniform, the probability corresponds to the ratio of the area representing the events and the total area.

a) It is a circle with the center at $(0,0)$ and radius 1, i.e. $\mathrm{P}\left(X^2 + Y^2 < 1\right) = \frac{\pi}{4}$

b) The area equals to the half of the square, i.e. $P(2X - Y > 0) = 0.5$

c) $P(|X + Y| < 2) = 1$

4.4

a) C is the normalization constant, i.e. $\int_0^1 \int_0^2 f(x,y)dxdy = 1$, which gives $C = \frac{1}{4}$

b)

$$f(x) = \begin{cases} \int_0^1 f(x,y)dy = \frac{x+1}{4} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

c)

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)dxdy = \begin{cases} 0 & x \leq 0 \text{ or } y \leq 0 \\ \frac{x^2 y}{8} + \frac{y^2 x}{4} & 0 < x < 2 \text{ and } 0 < y < 1 \\ \frac{y}{2} + \frac{y^2}{2} & 2 \leq x \text{ and } 0 < y < 1 \\ \frac{x^2}{8} + \frac{x}{4} & 0 < x < 2 \text{ and } 1 \leq y \\ 1 & 2 \leq x \text{ and } 1 \leq y \end{cases}$$

d) Theorem 2.1.5 gives $f_Z(z) = \frac{9}{8z^2}$, $1 < z < 9$.

4.10

a) $\mathrm{P}(X = x)\,\mathrm{P}(Y = y) \neq \mathrm{P}(X = x, Y = y)$

b)

		U		
		1	2	3
V	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

4.34

a)

$$P(X = x) = \int_0^1 P(X = x|p) f_P(p)dp = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}dp$$

$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{x+\alpha-1}(1-p)^{n+\beta-x-1}dp$$

using the definition of the beta function, $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, and the relation between the beta and gamma function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we get

$$P(X = x) = \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha+n+\beta)}.$$

b) $X|P \sim NegativeBinomial(r, P)$ and $P \sim Beta(\alpha, \beta)$.
Using the same approach as in a)

$$P(X = x) = \int_0^1 P(X = x|p) f_P(p)dp = \int_0^1 \binom{r+x+1}{x} p^r (1-p)^x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}dp$$

$$= \binom{r+x+1}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{r+\alpha-1}(1-p)^{x+\beta-1}dp =$$

$$= \binom{r+x+1}{x} \frac{\Gamma(\alpha+\beta)\Gamma(r+\alpha)\Gamma(x+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(r+x+\alpha+\beta)}$$

The mean and variance can be computed with use of the theorems 4.4.3 and 4.4.7, which gives

$$E(X) = E(E(X|P)) = E\left(\frac{r(1-P)}{P}\right) = \frac{r\beta}{\alpha-1}$$

$$\text{Var}(X) = E(Var(X|P)) + Var(E(X|P)) = E\left(\frac{r(1-p)}{p^2}\right) + Var\left(\frac{r(1-p)}{p}\right)$$

$$= \frac{r\beta(\alpha+\beta-1)}{(\alpha-1)(\alpha-2)} + \frac{r^2\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)}$$