TMA4295 Statistical inference
Exercise 6 - solution

Problem 5.36
$Y|N = n \sim \chi^2(2n)$ and $N \sim Poisson(\theta)$.

a)

$$E(Y) = E(E(Y|N)) = E(2N) = 2E(N) = 2\theta$$
$$Var(Y) = E(Var(Y|N)) + Var(E(Y|N)) = E(4N) + Var(2N) = 4\theta + 4\theta = 8\theta.$$

b) Let $Y = \sum_{i=1}^{\theta} X_i$, where $E(X_i) = 2$, $Var(X_i) = 8$. Then $E(Y) = 2\theta$ and $Var(Y) = 8\theta$ and

$$\frac{Y - E(Y)}{\sqrt{Var(Y)}} = \frac{Y - 2\theta}{\sqrt{8\theta}} = \frac{\theta\bar{X} - 2\theta}{\sqrt{8\theta}} = \frac{\sqrt{\theta}(\bar{X} - 2)}{\sqrt{8}} \longrightarrow N(0,1).$$

Problem 5.43

a) $\forall\epsilon > 0$ we get

$$P(|Y_n - \mu| > \epsilon) = P(Y_n - \mu > \epsilon) + P(Y_n - \mu < -\epsilon) = P(\sqrt{n}(Y_n - \mu) > \epsilon\sqrt{n}) + P(\sqrt{n}(Y_n - \mu) < -\epsilon\sqrt{n}).$$

Define $F_{Y_n}(t)$ the cdf of $\sqrt{n}(Y_n - \mu)$. Then

$$\lim_{n\to\infty} P(|Y_n - \mu| > \epsilon) = \lim_{n\to\infty}(1 - F_{Y_n}(\epsilon\sqrt{n}) + F_{Y_n}(-\epsilon\sqrt{n})) = \lim_{n\to\infty}(1 - \Phi(\epsilon\sqrt{n}/\sigma) + \Phi(-\epsilon\sqrt{n}/\sigma)) = 0.$$

Problem 5.44
$X_i \sim Bernoulli(p)$ iid, let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $E(X_i) = p$ and $Var(X_i) = p(1-p)$.

a) From the CLT we have

$$\frac{\sqrt{n}(Y_n - p)}{\sqrt{p(1-p)}} \longrightarrow N(0,1) \quad \text{in distribution}$$
$$\Rightarrow \sqrt{n}(Y_n - p) \longrightarrow N(0, p(1-p)) \quad \text{in distribution}$$

b) Here we want to use theorem 5.5.24, so let's define $g(Y_n) = Y_n(1 - Y_n)$, so $g(p) = p(1-p)$ and $g'(Y_n) = 1 - 2Y_n$ or $g'(p) = 1 - 2p \neq 0$ if $p \neq 1/2$. Using the theorem we get

$$\sqrt{n}\left[Y_n(1 - Y_n) - p(1-p)\right] \longrightarrow N\left(0, p(1-p)(1-2p)^2\right).$$

c) Here we want to use theorem 5.5.26. Let $g(Y_n) = Y_n(1 - Y_n)$, so $g(p) = p(1-p)$ and $g''(1/2) = -2$. Using the theorem we get

$$n\left[Y_n(1 - Y_n) - g(1/2)\right] \longrightarrow \sigma^2 \frac{g''(1/2)}{2}\chi^2(1)$$

which gives

$$n\left[Y_n(1 - Y_n) - 1/4\right] \longrightarrow -\frac{1}{4}\chi^2(1).$$

Problem 6.1
Using the factorization theorem (6.2.6) we can prove that $|X|$ is a sufficient statistics, since the pdf of X can be factorized as

$$f(x|\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}\sigma}e^{-|x|^2/2\sigma^2} = g(|X||\sigma) * 1.$$