Solution Exercise 9, TMA4295

7.2 a.

$$\begin{aligned} L(\beta|x) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta} \;=\; \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left[\prod_{i=1}^{n} x_i\right]^{\alpha-1} e^{-\Sigma_i x_i/\beta} \\ \log L(\beta|x) &= -\log\Gamma(\alpha)^n - n\alpha\log\beta + (\alpha-1)\log\left[\prod_{i=1}^{n} x_i\right] - \frac{\sum_i x_i}{\beta} \\ \frac{\partial \log L}{\partial\beta} &= -\frac{n\alpha}{\beta} + \frac{\sum_i x_i}{\beta^2} \end{aligned}$$

Set the partial derivative equal to 0 and solve for β to obtain $\hat{\beta} = \sum_i x_i/(n\alpha)$. To check that this is a maximum, calculate

$$\left.\frac{\partial^2 \log L}{\partial\beta^2}\right|_{\beta=\hat{\beta}} = \left.\frac{n\alpha}{\beta^2} - \frac{2\sum_i x_i}{\beta^3}\right|_{\beta=\hat{\beta}} = \frac{(n\alpha)^3}{\left(\sum_i x_i\right)^2} - \frac{2(n\alpha)^3}{\left(\sum_i x_i\right)^2} = -\frac{(n\alpha)^3}{\left(\sum_i x_i\right)^2} < 0.$$

Because $\hat{\beta}$ is the unique point where the derivative is 0 and it is a local maximum, it is a global maximum. That is, $\hat{\beta}$ is the MLE.

7.24 For n observations, $Y = \sum_i X_i \sim \text{Poisson}(n\lambda)$.

a. The marginal pmf of Y is

$$\begin{aligned} m(y) &= \int_0^\infty \frac{(n\lambda)^y e^{-n\lambda}}{y!} \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}\, d\lambda \\ &= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{(y+\alpha)-1} e^{-\frac{\lambda}{\beta/(n\beta+1)}}\, d\lambda \;=\; \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha) \left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}. \end{aligned}$$

Thus,

$$\pi(\lambda|y) = \frac{f(y|\lambda)\pi(\lambda)}{m(y)} = \frac{\lambda^{(y+\alpha)-1} e^{-\frac{\lambda}{\beta/(n\beta+1)}}}{\Gamma(y+\alpha)\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}} \sim \text{gamma}\left(y+\alpha, \frac{\beta}{n\beta+1}\right).$$

b.

$$\begin{aligned} \mathrm{E}(\lambda|y) &= (y+\alpha)\frac{\beta}{n\beta+1} \;=\; \frac{\beta}{n\beta+1} y + \frac{1}{n\beta+1}(\alpha\beta). \\ \mathrm{Var}(\lambda|y) &= (y+\alpha)\frac{\beta^2}{(n\beta+1)^2}. \end{aligned}$$

7.40

$$\begin{aligned}
\frac{\partial}{\partial \theta}\log L(\theta|\mathbf{x}) &= \frac{\partial}{\partial p}\log \prod_i p^{x_i}(1-p)^{1-x_i} = \frac{\partial}{\partial p}\sum_i x_i \log p + (1-x_i)\log(1-p) \\
&= \sum_i \left[\frac{x_i}{p} - \frac{(1-x_i)}{1-p}\right] = \frac{n\bar{x}}{p} - \frac{n-n\bar{x}}{1-p} = \frac{n}{p(1-p)}[\bar{x}-p].
\end{aligned}$$

By Corollary 7.3.15, $\bar{X}$ is the UMVUE of p and attains the Cramér-Rao lower bound. Alternatively, we could calculate

$$\begin{aligned}
&-n\mathrm{E}_\theta\left(\frac{\partial^2}{\partial \theta^2}\log f(X|\theta)\right) \\
&= -n\mathrm{E}\left(\frac{\partial^2}{\partial p^2}\log\left[p^X(1-p)^{1-X}\right]\right) = -n\mathrm{E}\left(\frac{\partial^2}{\partial p^2}\left[X\log p + (1-X)\log(1-p)\right]\right) \\
&= -n\mathrm{E}\left(\frac{\partial}{\partial p}\left[\frac{X}{p} - \frac{(1-X)}{1-p}\right]\right) = -n\mathrm{E}\left(\frac{-X}{p^2} - \frac{1-X}{(1-p)^2}\right) \\
&= -n\left(-\frac{1}{p} - \frac{1}{1-p}\right) = \frac{n}{p(1-p)}.
\end{aligned}$$

7.41 a. $\mathrm{E}\left(\sum_i a_i X_i\right) = \sum_i a_i \mathrm{E}\,X_i = \sum_i a_i\mu = \mu\sum_i a_i = \mu$. Hence the estimator is unbiased.

b. $\mathrm{Var}\left(\sum_i a_i X_i\right) = \sum_i a_i^2 \mathrm{Var}\,X_i = \sum_i a_i^2\sigma^2 = \sigma^2\sum_i a_i^2$. Therefore, we need to minimize $\sum_i a_i^2$, subject to the constraint $\sum_i a_i = 1$. Add and subtract the mean of the a_i, $1/n$, to get

$$\sum_i a_i^2 = \sum_i\left[\left(a_i - \frac{1}{n}\right) + \frac{1}{n}\right]^2 = \sum_i\left(a_i - \frac{1}{n}\right)^2 + \frac{1}{n},$$

because the cross-term is zero. Hence, $\sum_i a_i^2$ is minimized by choosing $a_i = 1/n$ for all i. Thus, $\sum_i (1/n)X_i = \bar{X}$ has the minimum variance among all linear unbiased estimators.

Problem 5
$X_1, ..., X_n$ i.i.d. $N(\theta, \sigma^2)$.

a) The normal pdf satisfies the assumption of the Cramer-Rao theorem and lemma 7.3.11 So we have

$$\frac{\partial^2}{\partial\theta^2} \log\left(\frac{1}{(2\pi\sigma^2)^{-1/2}} \exp\left(-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}\right)\right) = \frac{\partial^2}{\partial\theta^2}\left(-\frac{1}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}\right) = -\frac{1}{\sigma^2}$$

and

$$-E\left(\frac{\partial^2}{\partial\theta^2}\log f(X)|\theta\right) = \frac{1}{\sigma^2}.$$

Thus the lower bound is given by $\frac{\sigma^2}{n}$.

b) $\bar{X}$ is UMVUE since $E(\bar{X}) = \theta$ and $Var(\bar{X}) = \frac{\sigma^2}{n}$.

c) Using lemma 7.3.11 we have

$$\frac{\partial^2}{\partial(\theta^2)^2} \log\left(\frac{1}{(2\pi\sigma^2)^{-1/2}} \exp\left(-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}\right)\right) = \frac{\partial^2}{\partial(\theta^2)^2}\left(-\frac{1}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2}\frac{x^2+\theta^2-2x\sqrt{\theta^2}}{\sigma^2}\right) = -\frac{x}{4\sigma^2\theta^3}$$

end

$$-E\left(\frac{\partial^2}{\partial(\theta^2)^2}\log f(X)|\theta^2\right) = \frac{\theta}{4\sigma^2\theta^3} = \frac{1}{4\sigma^2\theta^2}.$$

Thus the lower bound is given by $\frac{4\sigma^2\theta^2}{n}$.

d) $W(\mathbf{X})$ is unbiased since

$$E(\mathbf{X}) = E\left(\bar{X^2} - \frac{\sigma^2}{n}\right) = E(\bar{X^2}) - \frac{\sigma^2}{n} = \frac{1}{n}n(\theta^2+\sigma^2) - \frac{\sigma^2}{n} = \theta^2,$$

but the variance

$$Var(W(\mathbf{X})) = Var(\bar{X^2}) = \frac{1}{n^2}nVar(X_i) = \frac{1}{n}\left(E(X_i^4) - E(X_i^2)^2\right) = \frac{1}{n}(4\theta^2\sigma^2 + 2\sigma^4)$$

Problem 6
$X_1, ..., X_n$ i.i.d. $N(\mu, \sigma^2)$ with μ and σ^2 unkwown. Using lemma 7.3.11 as we did in the previous exercise we can see that the lower bound on the variance for unbiased estimators of μ is given by $\frac{\sigma^2}{n}$ and the lower bound on the variance for unbiased estimators of σ^2 is given by $\frac{2\sigma^4}{n}$.
The variance of $\bar{x}$ is given by

$$Var(\bar{X}) = \frac{1}{n^2}nVar(x_i) = \frac{\sigma^2}{n}.$$

The variance of S^2 is given by

$$Var(S^2) = \frac{2\sigma^4}{n-1}.$$