Exercise set 8

- Many use very complicated methods for finding the asymptotic variance of S_n in problem 2c). It is, however, enough to notice that

$$\sum_i \frac{(Y_i - \mu)^2}{\sigma^2} \overset{d}{=} Z_n,$$

meaning that S_n is equal in distribution to W_n times some constants,

$$S_n \overset{d}{=} \sigma 2^{1/4} W_n$$

Since the asymptotic distribution of W_n is $N(2^{-1/4}, \frac{2^{-3/2}}{n})$, the asymptotic distribution of S_n is the asymptotic distribution of $\sigma 2^{1/4} W_n$ which is $N(\sigma, \frac{\sigma^2}{2n})$. It is not necessary to use the delta-method once more, since we already did that for finding the asymptotic distribution of W_n, and it is not necessary to use the delta-method for finding the asymptotic distribution of $c \cdot W_n$, since c is only a constant, and we know how normally distributed variables behave when we multiply them by constants.

- The distribution of a maximum is not the same as the distribution of a random variable. We can therefore not say that $E[\max X] = \theta/2$. Use

$$P(\max X \leq x) = P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) = P(X_1 \leq x)^n,$$

when all random variables are iid.

- In problem 3c) a lot of people forget to define where the likelihood is non-zero. Most people say that

$$L(\theta) = \theta^{-n},$$

but then the maximum likelihood estimator would be $\hat{\theta} = 0$, not $\hat{\theta} = \max X$. The only way to find the MLE is to use the correct likelihood function

$$L(\theta) = \theta^{-n} \prod_{i=1}^{n} I_{[0,\theta]}(X_i) = \theta^{-n} I_{(-\infty,\theta]}(\max X) I_{[0,\infty)}(\min X).$$

- In problem 4c) most people comment that we put more weight on the prior if σ^2 is larger than τ^2, and more weight on the observations if τ^2 is larger than σ^2, but almost none mention the fact that we have an n in the expression as well. The weights are σ^2 and $n\tau^2$, meaning that we also put more weight on the observations if we have a lot of observations, and more weight on the prior if we have few observations.

- In problem 2b) it is not necessary to prove that the distribution of Z_n is a gamma-distribution using moment-generating functions. This is a well-known thing by now, and the exercise only asks us what the distribution is. It is enough to say that we have a sum of iid gamma-distributed variables, and we know that the sum of iid gamma-distributed variables is gamma-distributed and what the parameters in the distribution are.