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- Given a distribution $f(x)$
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- Are you able to generate samples from any distribution $f(x)$?

Recall: Bayesian example

- A simple example (from George et al., 1993)
  - Analysis of 10 power plant pumps
  - x_i, t_i: number of failures for pump i and length of operation time on that pump (in 1000 hours)
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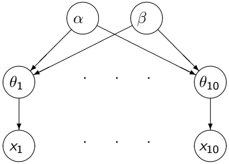

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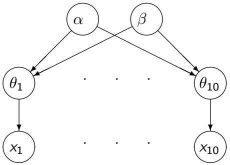

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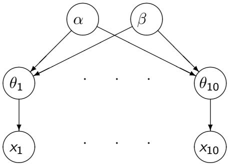

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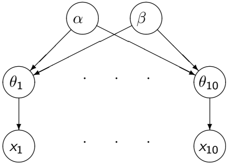
$$f(\alpha, \beta, \theta_1, \ldots, \theta_{10}|x_1, \ldots, x_{10})$$

Posterior distribution of interest

- Graphical model:



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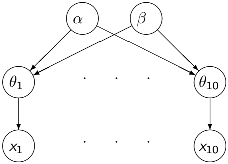

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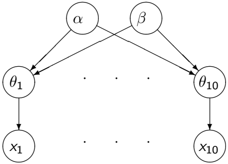

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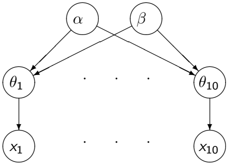

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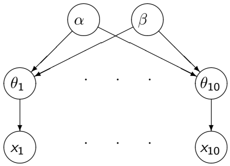

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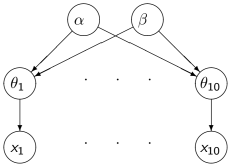

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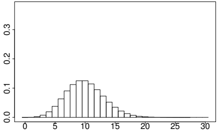
- Can you generate samples from $f(\alpha, \beta, \theta_1, \ldots, \theta_{10} | x_1, \ldots, x_{10})$?

A (very) simple MCMC example

- Note: This is just for illustration, you should never never use MCMC for this distribution!
- Let

$$f(x) = \frac{10^x}{x!} e^{-10} \ , \ \ x = 0, 1, 2, \ldots$$



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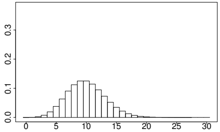


- Set x_0 to 0, 1 or 2 with probability 1/3 for each
- Markov chain kernel

$$\mathrm{P}(x_{i+1} = x_i - 1 | x_i) = \begin{cases} x_i/20 & \text{if } x_i \leq 9, \\ 1/2 & \text{if } x_i > 9 \end{cases}$$

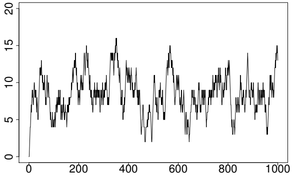
$$\mathrm{P}(x_{i+1} = x_i | x_i) = \begin{cases} (10 - x_i)/20 & \text{if } x_i \leq 9, \\ (x_i - 9)/(2(x_i + 1)) & \text{if } x_i > 9 \end{cases}$$

$$\mathrm{P}(x_{i+1} = x_i + 1 | x_i) = \begin{cases} 1/2 & \text{if } x_i \leq 9, \\ 5/(x_i + 1) & \text{if } x_i > 9 \end{cases}$$

- This Markov chain has limiting distribution $f(x)$
  - will explain why later

A (very) simple MCMC example (cont.)

- Trace plot:



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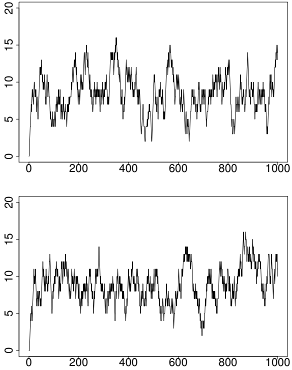

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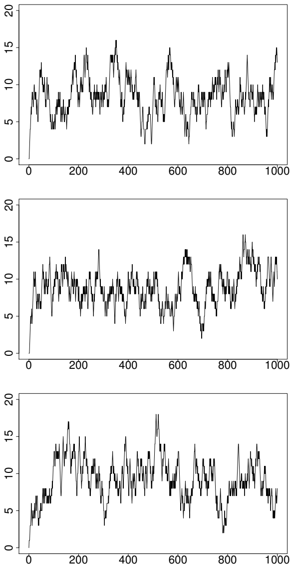

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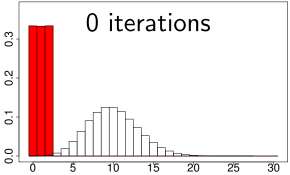

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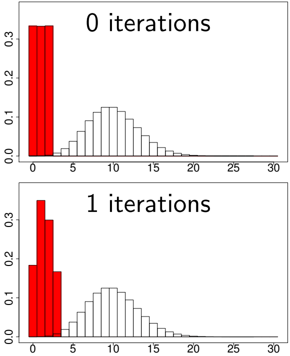

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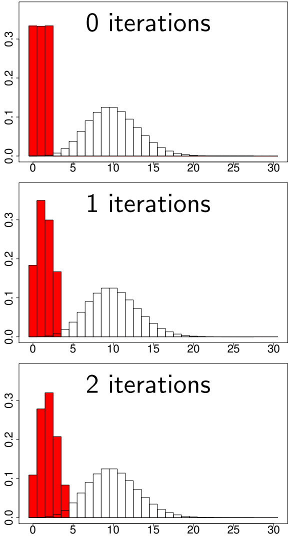

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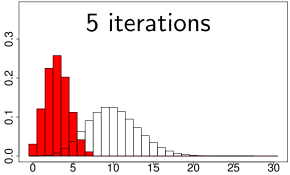

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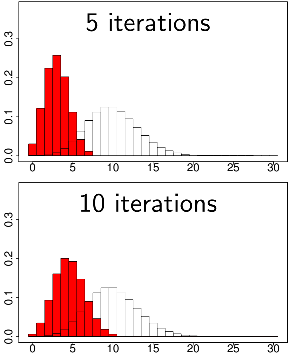

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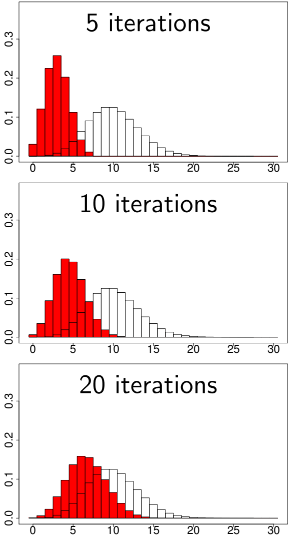

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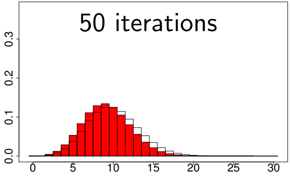

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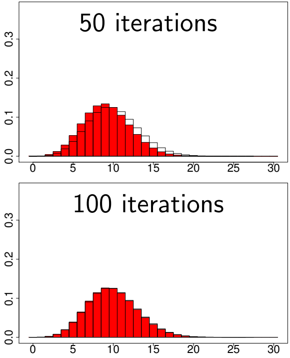

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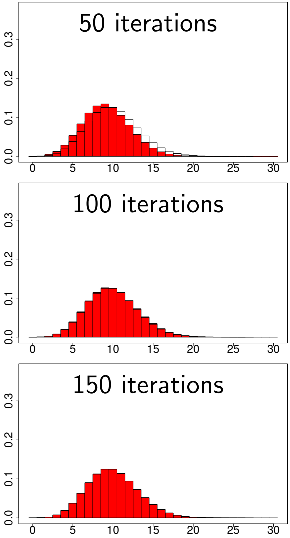

Interactive lecture tomorrow

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- Problem B: You are given a $f(x), x \in \{1, 2, 3\}$.
  - find one transition matrix with limiting distribution $f(x)$
  - find another transition matrix with limiting distribution $f(x)$
  - define a transition matrix as a linear combination of the two above. What is the limiting distribution of this new Markov chain?