Bootstrapping

⋆ Schematic view of bootstrapping

Real world

$$\begin{array}{ccc} P & \longrightarrow & x = (x_1, \ldots, x_n) \\ & & \downarrow \\ & & \widehat{\theta} = s(x) \end{array}$$

$$\Longrightarrow$$

Bootstrap world

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Real world $\qquad\qquad\qquad\qquad$ Bootstrap world

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- ⋆ We have discussed bootstrapping for estimating
  - standard deviation
  - bias

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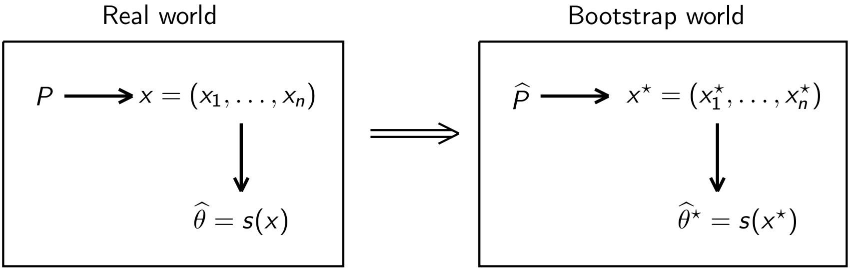

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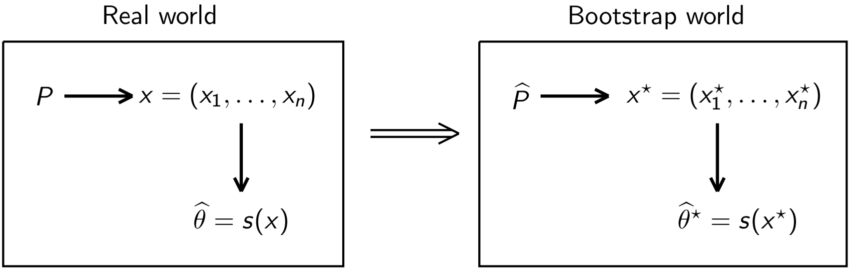

- ⋆ We have discussed bootstrapping for estimating
  - standard deviation
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- ⋆ We start by defining the ideal bootstrap estimator
  - estimate it via Monte Carlo sampling
- ⋆ Today: bias-correction, confidence intervals, permutation tests

Bootstrapping of bias

- Observed data: $x_1, \ldots, x_n \sim F$ iid
- Parameter of interest: $\theta = t(F)$
- Estimator: $\widehat{\theta} = s(x)$
- Bias of $\widehat{\theta}$:
$$\text{bias}_F = \text{bias}_F(\widehat{\theta}, \theta) = E_F[s(x)] - t(F)$$
- Ideal bootstrap estimator for bias:
$$\text{bias}_{\widehat{F}} = E_{\widehat{F}}[s(x^\star)] - t(\widehat{F})$$
- boostrap estimate for the bias:
$$\widehat{\text{bias}}_B = \widehat{\theta}^\star(\cdot) - t(\widehat{F})$$

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- Note: $\text{Var}[\widehat{\theta}_c] \geq \text{Var}[\widehat{\theta}]$ so bias correction will not always give a better estimator. If the bias is small we should not do bias correction.

Recall: Derivation of the t-interval

- ⋆ Assume: $x_1, \ldots, x_n \sim F$ (iid)
  - notation: $\mathrm{E}[x_i] = \theta$, $\mathrm{Var}[x_i] = \sigma^2$
- ⋆ Estimator for θ:

$$\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx N\left(\theta, \mathrm{se}^2\right) \quad \text{where} \quad \mathrm{se}^2 = \mathrm{Var}\left[\widehat{\theta}\right] = \frac{\sigma^2}{n}$$

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- Confidence interval

$$\left[\widehat{\theta} - t_{\frac{\alpha}{2},n-1}\widehat{\mathrm{se}}, \widehat{\theta} + t_{\frac{\alpha}{2},n-1}\widehat{\mathrm{se}}\right]$$

Recall: The z-confidence interval

- The z-confidence interval (assuming n to be large):

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  - if we draw a sample $\theta \sim N\left(\widehat{\theta}, \widehat{\mathrm{se}}\right)$
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- Note: This interpretation is ideal for bootstrapping