Introduction to Bayesian statistics

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  - A billiard ball is dropped on the interval $[0, 1]$
    - it stops at p
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- This is a beta-distribution, $\mathcal{B}(x+1, n-x+1)$, with

$$\mathrm{E}[p|x] = \frac{x+1}{n+2}$$

- A natural estimator for p

$$\widehat{p} = \frac{X+1}{n+2}$$

Bayesian statistics

- In previous example: p is a stochastic variable because it is the result of a stochastic experiment
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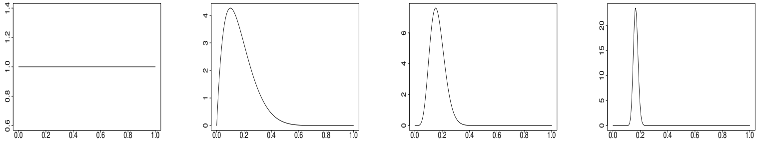

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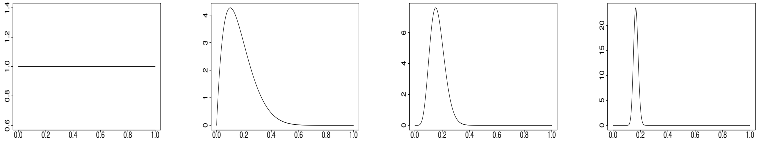

- we roll the dice n times, let x: number of sixes

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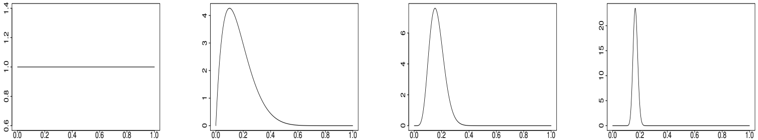
- Thus $p|x \sim \mathcal{B}(\alpha + x, \beta + n - x)$, so
$$E[p|x] = \frac{\alpha + x}{\alpha + \beta + n}$$
- Observed $n = 100$, $x = 26$:

Bayesian statistics — an example

- Before observing the value of x, $f(p)$:



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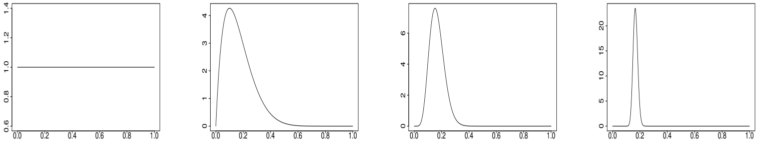

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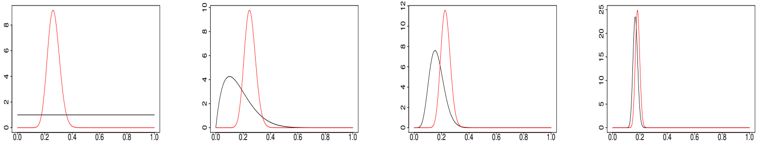

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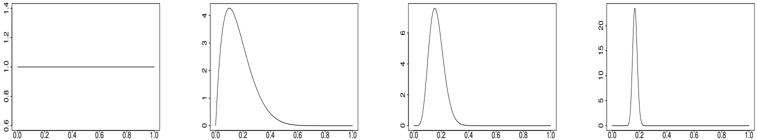

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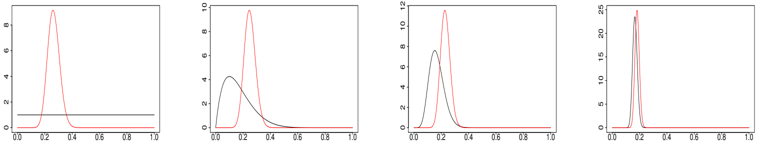

$E[p|x] = 0.265$ $\quad$ $E[p|x] = 0.255$ $\quad$ $E[p|x] = 0.230$ $\quad$ $E[p|x] = 0.183$

Interpretation of probability

- Frequentist (objective): Probability of event A is

$$\mathrm{P}(A) = \lim_{n \to \infty} \frac{m}{n}$$

  where m: # times A occurres in n identical and independent trials

- Bayesian (subjective): Probability of event A, $\mathrm{P}(A)$, is a measure of someone's degree of belief in the occurrence of A.
  - different persons may have different $\mathrm{P}(A)$

Prior and posterior distribution

- Prior distribution: $f(\theta)$
  - a measure of our belief about the value of θ before we have observed the data, based on prior information/experience
- Observation and Likelihood: $f(x|\theta)$
  - observed value x, and its probability distribution given θ
- Posterior distribution: $f(\theta|x)$
  - a measure of our belief about the of value of θ after we have observed the data x, based on prior information/experience *and* the observed data x
  - Bayes theorem

$$f(\theta|x) = \frac{f(\theta, x)}{f(x)} \propto f(\theta, x) = f(\theta)f(x|\theta)$$

Conjugate priors

- In examples: posteriors are all available on closed form
  - this is because we have used *conjugate* priors
- binomial conjugate prior
  - $x|p \sim \text{binomial}(n, p)$
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- normal (mean) conjugate prior
  - $x_1, \ldots, x_n|\mu \sim \text{N}(\mu, \sigma_0^2)$
  - $\mu \sim \text{N}(\mu_0, \tau^2)$
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- normal (variance) conjugate prior
  - $x_1, \ldots, x_n|\sigma^2 \sim \text{N}(\mu_0, \sigma^2)$
  - $\sigma^2 \sim \text{IG}(\alpha, \beta)$
  - $\sigma^2|x_1, \ldots, x_n \sim \text{IG}(\cdot, \cdot)$

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- Conjugate priors makes analytical evaluations easier
  - and may make sampling from the posterior easier ...

Hierarchical Bayesian modeling — a simple example

- A simple example (from George et al., 1993)
  - Analysis of 10 power plant pumps
  - x_i, t_i: number of failures for pump i and length of operation time on that pump (in 1000 hours)
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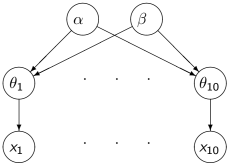

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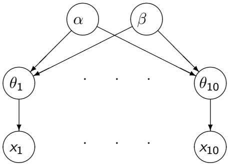

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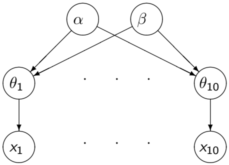


  - observed: $x_1, \ldots, x_n$
  - posterior distribution of interest:

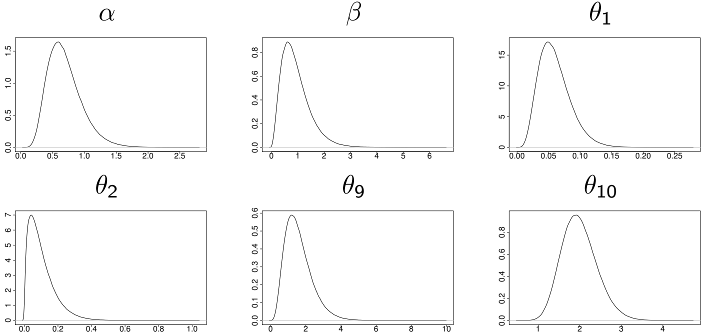
$$f(\alpha, \beta, \theta_1, \ldots, \theta_{10}|x_1, \ldots, x_{10})$$

Hierarchical Bayesian modeling — a simple example

- Data:

Pump	1	2	3	4	5	6	7	8	9	10
t_i	94.3	15.7	62.9	126	5.24	31.4	1.05	1.05	2.1	10.5
x_i	5	1	5	14	3	19	1	1	4	22

- Posterior density plots:



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- Posterior mean for θ_i compared to x_i/t_i

parameter	posterior mean	x_i/t_i
θ_1	0.0598	0.0530
θ_2	0.1017	0.0636
θ_3	0.0892	0.0795
θ_4	0.1157	0.1111
θ_5	0.6011	0.5725
θ_6	0.6095	0.6051
θ_7	0.8910	0.9524
θ_8	0.8928	0.9524
θ_9	1.5867	1.9047
θ_{10}	1.9901	2.0952