DERIVATION UNDER THE INTEGRAL SIGN

In this note we discuss some very useful results concerning derivation of integrals with respect to a parameter, as in

$$\int_V f(x,y)\,dy,$$

where x is the parameter. All the results we give here are based on the ***dominated convergence theorem***, which is a very strong theorem about convergence of integrals. This theorem is one of the highlights of the modern theory of integration (the Lebesgue theory). However, it is not necessary to know this theory to use the theorem: Nothing stops us from applying it to the Riemann integral which we know from calculus.

Throughout we assume $V \subset \mathbb{R}^n$ is an open set, and $L^1(V)$ denotes the set of (Riemann) integrable functions $f : V \to \mathbb{R}$.

Theorem A. *Suppose we are given a sequence of functions $F_j(y)$ ($j \in \mathbb{N}$, $y \in V$) such that*

(i) *For every $y \in V$, $\lim_{j\to\infty} F_j(y) =: F(y)$ exists.*
(ii) *There exists $g \in L^1(V)$, $g \geq 0$, such that $\left|F_j(y)\right| \leq g(y)$ for all $j \in \mathbb{N}$ and $y \in V$.*

Under these assumptions,

$$\lim_{j\to\infty} \int_V F_j(y)\,dy = \int_V F(y)\,dy.$$

Needless to say, we do not prove this theorem here, but we shall apply it to prove the following theorems.

Theorem B. *(Continuity with respect to a parameter.) Consider*

$$\phi(x) = \int_V f(x,y)\,dy \qquad (x \in U),$$

where $U \subset \mathbb{R}^m$ is open and $f : U \times V \to \mathbb{R}$ satisfies the following assumptions:

(i) *For every $y \in V$, the function $x \mapsto f(x,y)$ is continuous on U.*
(ii) *There exists $g \in L^1(V)$, $g \geq 0$, such that $\left|f(x,y)\right| \leq g(y)$ for all $x \in U$, $y \in V$.*

Then $\phi \in C(U)$.

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Proof. It suffices to prove that if x_j is a sequence in U, and if $x_j \to x$ as $j \to \infty$, where $x \in U$, then also $\phi(x_j) \to \phi(x)$. But this follows from Theorem A applied to $F_j(y) = f(x_j, y)$ and $F(y) = f(x, y)$. □

Theorem C. *(Derivation under the integral sign.) Consider*

$$\phi(t) = \int_V f(t, y)\, dy \qquad (t \in [a, b]),$$

where $f : [a, b] \times V \to \mathbb{R}$ *satisfies the following assumptions:*

(i) $f_t(t, y)$ *exists for all* $a \le t \le b$, $y \in V$.
(ii) *There exists* $g \in L^1(V)$, $g \ge 0$, *such that* $\left|f_t(t, y)\right| \le g(y)$ *for all* $a \le t \le b$, $y \in V$.

Then ϕ *is differentiable, and*

$$\phi'(t) = \int_V f_t(t, y)\, dy \qquad (t \in [a, b]).$$

Proof. It suffices to prove that if t_j is a sequence in $[a, b]$, and if $t_j \to t$ as $j \to \infty$, then

$$\lim_{j\to\infty} \frac{\phi(t_j) - \phi(t)}{t_j - t} = \int_V f_t(t, y)\, dy.$$

We apply Theorem A with

$$F_j(y) = \frac{f(t_j, y) - f(t, y)}{t_j - t}$$

and $F(y) = f_t(x, y)$. We must check that the hypotheses of Theorem A are satisfied. By assumption (i), $\lim_{j\to\infty} F_j(y) = F(y)$ for all y, so (i) in Theorem A is satisfied. By the mean value theorem,

$$\frac{f(t_j, y) - f(t, y)}{t_j - t} = f_t(t_j^*, y)$$

for some t_j^* (depending on y) between t and t_j. Thus, assumption (ii) implies

$$\left|F_j(y)\right| \le g(y)$$

for all j and all y, so (ii) in Theorem A is also satisfied. □

An important remark is that Theorem C can be applied to an integral depending on multiple parameters,

$$\phi(x) = \int_V f(x, y)\, dy$$

where $x \in U \subset \mathbb{R}^m$, by considering each component x_i separately. So if $f_{x_i}(x, y)$ exists and $\left|f_{x_i}(x, y)\right| \le g(y)$ for all x, y and $i = 1, \ldots, m$, where

$g \in L^1(V)$, then

$$\phi_{x_i}(x) = \int_V f_{x_i}(x, y)\, dy$$

for $x \in U$ and $i = 1, \ldots, m$.

Finally, here is a theorem about differentiation of an integral where the parameter occurs both in the integration limits and in the integrand.

Theorem D. *Consider*

$$\phi(t) = \int_0^t f(t, s)\, ds \qquad (0 < t < b)$$

where $f : (0, b) \times (0, b) \to \mathbb{R}$ *satisfies the following assumptions:*

(i) $f_t(t, s)$ *exists for all* $0 < s, t < b$.
(ii) f *and* f_t *are continuous on* $(0, b) \times (0, b)$.
(iii) *There exists* $g \in L^1(0, b)$, $g \geq 0$, *such that*

$$\left|f(t, s)\right|, \left|f_t(t, s)\right| \leq g(y) \qquad \textit{for all } 0 < s, t < b.$$

Then $\phi \in C^1\big((0, b)\big)$ *and*

$$\phi'(t) = f(t, t) + \int_0^t f_t(t, s)\, ds \qquad (0 < t < b).$$

Proof. Define

$$\psi(t, \tau) = \int_0^\tau f(t, s)\, ds \qquad (0 < t, \tau < b).$$

Theorem C implies (here we use assumptions (i) and (iii))

$$\partial_t \psi = \int_0^\tau f_t(t, s)\, ds.$$

The fundamental theorem of calculus implies (here we use assumption (ii) on f)

$$\partial_\tau \psi = f(t, \tau).$$

Theorem B implies that $\partial_t \psi$ is continuous (here we use assumptions (ii) and (iii)). We conclude that

$$\psi \in C^1\big((0, b) \times (0, b)\big),$$

and since $\phi(t) = \psi(t, t)$, the chain rule implies that $\phi \in C^1\big((0, b)\big)$ and

$$\phi'(t) = \partial_t \psi(t, t) + \partial_\tau \psi(t, t) = \int_0^t f_t(t, s)\, ds + f(t, t).$$

□