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TMA4305 Partial Differential Equations Spring 2006

Exercise set 1

Throughout, U denotes an open, bounded subset of $\mathbb{R}^n$, with C^1 boundary ∂U.

$\boxed{1}$ Consider the *Neumann problem*

$$(*)\qquad \begin{cases} \Delta u = f & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial U, \end{cases}$$

where $f \in C(U)$ is given.

a) What can we surely add to any solution to get another solution? Conclude that we don't have uniqueness for $(*)$.

b) Show that

$$\int_U f(x)\,dx = 0$$

is a necessary condition for $(*)$ to have a solution $u \in C^2(U) \cap C^1\big(\overline{U}\big)$.

c) Can you give a physical interpretation of part (b), for stationary heat flow with source f?

$\boxed{2}$ Consider the Laplace equation with a Neumann boundary condition:

$$(*)\qquad \begin{cases} \Delta u = 0 & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial U, \end{cases}$$

where $g \in C(\partial U)$ is given.

a) Prove that

$$\int_{\partial U} g(x)\,dS(x) = 0$$

is a necessary condition for $(*)$ to have a solution $u \in C^2(U) \cap C^1\big(\overline{U}\big)$.

b) Prove Dirichlet's principle for $(*)$. It asserts that $u \in C^2(U) \cap C^1\big(\overline{U}\big)$ solves $(*)$ if and only if it is a minimum for the energy

$$I[w] = \frac{1}{2}\int_U |\nabla w|^2\,dx - \int_{\partial U} g\,w\,dS,$$

where $w \in C^2(U) \cap C^1\big(\overline{U}\big)$ (note that there is no assumption on the boundary values of w!).

$\boxed{3}$ **a)** Show that the general solution of the PDE $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F, G.

b) Using the change of variables $\xi = x+t$, $\eta = x-t$, show that $u_{tt} = u_{xx}$ if and only if $u_{\xi\eta} = 0$.

c) Use (a) and (b) to rederive d'Alembert's formula.

4 Show that for a radial function $u(x,t) = u(r,t)$, where $r = |x|$ and $x \in \mathbb{R}^n$, the wave equation takes the form

$$u_{tt} = u_{rr} + \frac{n-1}{r} u_r.$$

5 The purpose of this exercise is to show that, among all possible dimensions, only in three space dimensions can one have distortionless radial wave propagation with attenuation. This means the following: A radial wave in n dimensions satisfies (see the previous problem)

$$(*) \qquad u_{tt} = u_{rr} + \frac{n-1}{r} u_r.$$

Consider such a wave that has the special form

$$(**) \qquad u(r,t) = \alpha(r) f\big(t - \beta(r)\big),$$

where f is a given function, called the *wave profile*, $\alpha(r)$ is called the *attenuation* and $\beta(r)$ the *delay*. The question is whether such solutions exist for "arbitrary" wave profiles f.

a) Plug $(**)$ into $(*)$ to get an ODE for f.

b) Since f is supposed to be arbitrary, conclude that the coefficients of f, f' and f'' must all equal zero. Solve the resulting ODEs to see that $n = 1$ or $n = 3$ (unless u vanishes identically).

c) If $n = 1$, show that $\alpha(r) = \text{const.}$ (so there is no attenuation).

6 Solve

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R},\ t > 0, \\ u(x,0) = x^2 & x \in \mathbb{R}, \end{cases}$$

by first showing that u_{xxx} must vanish.

7 Use the energy method to prove uniqueness of the initial/boundary value problem

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial U \times [0,T], \\ u = h & \text{on } U \times \{t = 0\}, \end{cases}$$

where we use the *Neumann boundary condition.*

8 Solve the heat equation with constant dissipation:

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c > 0$ is a constant. (*Hint:* Change variables to $v(x,t) = e^{ct} u(x,t)$.)

$\boxed{9}$ Solve the heat equation with variable dissipation:

$$\begin{cases} u_t - \Delta u + ct^2 u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c > 0$ is a constant. (*Hint:* To find a suitable change of variables, look at the solutions of the ODE $f'(t) + ct^2 f(t) = 0$.)

$\boxed{10}$ Solve the heat equation with convection:

$$\begin{cases} u_t - \Delta u + b \cdot D_x u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $b \in \mathbb{R}^n$ is a constant. (*Hint:* Motivated by the solution of the transport equation $v_t + b \cdot D_x v = 0$, change variables to $y = x - tb$.)

$\boxed{11}$ Find the radial solutions $u(r) = u(x)$, where $r = |x|$, of

$$\Delta u = ku \qquad \text{on } \mathbb{R}^3,$$

where $k > 0$ is a constant. (*Hint:* Look at $v = ru$.)

$\boxed{12}$ The purpose of this problem is to find a fundamental solution $\Phi(x)$ for $-\Delta = -\frac{d^2}{dx^2}$ on $\mathbb{R}$. To deduce the form of Φ, one can argue as follows:

$$\begin{aligned} & -\Phi'' = \delta && (\delta = \text{Dirac's delta function}) \\ \Longrightarrow\ & -\Phi' = H + C && (H = \text{Heaviside function}) \\ \Longrightarrow\ & -\Phi(x) = x_+ + Cx + D \end{aligned}$$

where

$$x_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and C, D are arbitrary constants.

a) Prove that if $f \in C_c(\mathbb{R})$, then the function

$$u(x) = \Phi * f(x) = \int_{-\infty}^{\infty} f(y)\Phi(x-y)\,dy \qquad (x \in \mathbb{R})$$

satisfies $-u'' = f$. Thus, Φ is a fundamental solution for $-\frac{d^2}{dx^2}$.

b) Show that in order to make $\Phi(-x) = \Phi(x)$, we must take $C = -1/2$, in which case

$$\Phi(x) = -\frac{|x|}{2} + D.$$

Which value of D makes Φ homogeneous?

$\boxed{13}$ Find the Green's function for the interval $(0, l)$, recalling that $\Phi(x) = -|x|/2$ is a fundamental solution for $-\frac{d^2}{dx^2}$.

14 Use the Green's function found in the previous problem to find a formula

$$u(x) = \int_0^l G(x,y) f(y)\, dy$$

for the solution of

$$\begin{cases} -u'' = f & \text{in } (0,l), \\ u(0) = u(l) = 0. \end{cases}$$

Prove by calculating u'' that the boundary value problem really is satisfied, if $f \in C([0,l])$.

15 Find Green's function for a half-ball $|x| < 1$, $x_n > 0$. (*Hint:* Use the Green's function for the whole ball and reflect across $x_n = 0$.)

16 Find Green's function for the octant

$$U_+ = \left\{x \in \mathbb{R}^3 : |x| < 1 \text{ and } x_1, x_2, x_3 > 0\right\}$$

of the unit ball $U = B(0,1)$ in $\mathbb{R}^3$.

17 One way to arrive at the form of the energy for a solution the wave equation $u_{tt} - \Delta u = 0$ in $U \times (0,\infty)$ is to multiply the equation by u_t and integrate over U, and then to integrate by parts, assuming that $u = 0$ on $\partial U \times [0,\infty)$. This gives

$$0 = \int_U u_t(u_{tt} - \Delta u)\, dx = \int_U \frac{1}{2}\frac{\partial}{\partial t}(u_t)^2 + \nabla u_t \cdot \nabla u\, dx = \frac{d}{dt}\left(\frac{1}{2}\int_U (u_t)^2 + |\nabla u|^2\, dx\right),$$

and the quantity inside the parentheses is the energy.

a) Use this idea to find the energy for a solution of the *Klein-Gordon equation* (appearing in quantum mechanics)

$$u_{tt} - \Delta u + u = 0.$$

b) Prove finite speed of propagation for the Klein-Gordon equation. (*Hint:* Modify the proof of Theorem 6 in §2.4.)

18 Solve the wave equation in 3d (i.e., $n = 3$) with initial data $u(x,0) = 0$, $u_t(x,0) = x_2$, by using Kirchoff's formula.

19 Solve the wave equation in 3d with initial data $u(x,0) = 0$, $u_t(x,0) = |x|^2$.

20 In which region does a 3d wave (i.e., a solution of the wave equation in 3d) certainly vanish, given that its initial data vanish outside a ball $B(0,r)$?

21

a) Find the area of that part of the sphere $\partial B(x,R)$ in $\mathbb{R}^3$ which lies inside the ball $B(0,\rho)$, where $x \in \mathbb{R}^3$ and $R, \rho > 0$. I.e., find the area of

$$\partial B(x,R) \cap B(0,\rho) \subset \mathbb{R}^3$$

as a function of $|x|$, R and ρ. (*Hint:* Use the law of cosines. Consider separately the cases $|x| \le \rho$ and $|x| > \rho$.)

b) Solve in 3d:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{for } t > 0, \\ u = 0,\ u_t = h & \text{at } t = 0, \end{cases}$$

where $h(x) = 1$ if $|x| \leq 1$ and $h(x) = 0$ if $|x| > 1$. Use Kirchoff's formula. (The solution will be continuous, but will have discontinuous derivatives, so it will be a "weak solution", but this small matter need not concern us here.)

c) Sketch the graph of u versus $|x|$, for $t = 1/2$, 1 and 2. This is a "movie" of the solution.

d) Sketch the graph of u versus t, for $|x| = 1/2$ and 2. This is what a stationary observer at these points in space would see, as time passes.

e) Let $|x_0| < 1$. Ride the wave along a "light ray" emanating from x_0, i.e., follow the curve parametrized by $x = x_0 + tv$, as t increases, where $v \in \mathbb{R}^3$ (the velocity) is a unit vector ($|v|$ = 1). Prove that

$$tu(x_0 + tv, t) \qquad \text{converges as } t \to \infty.$$