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TMA4305 Partial Differential Equations Spring 2006

Problem set 2

Unless stated otherwise, U always denotes an open, bounded subset of $\mathbb{R}^n$, with C^∞ boundary ∂U, and L is a symmetric, uniformly elliptic 2^{nd} order differential operator in divergence form, with smooth coefficients:

$$Lu = -\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j},$$

where

(i) (smoothness) $a_{ij} \in C^\infty(\overline{U})$,

(ii) (symmetry) $a_{ij} = a_{ji}$ and

(iii) (uniform ellipticity) for some $\theta > 0$:

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \theta\,|\xi|^2 \qquad \text{for all } x \in U \text{ and } \xi \in \mathbb{R}^n.$$

$\boxed{1}$ Suppose $u \in H_0^1(U)$, $f \in L^2(U)$, and that

(1) $$\int_U \nabla u \cdot \nabla v\,dx = \int_U f v\,dx$$

for all $v \in C_c^\infty(U)$. Prove that (1) then holds for all $v \in H_0^1(U)$.

$\boxed{2}$ Suppose $F : \mathbb{R} \to \mathbb{R}$ is C^1 with bounded derivative:

$$|F'| \le M.$$

Let $1 \le p < \infty$ ($p < \infty$ is not really necessary, but we assume it for simplicity). Suppose $u \in W^{1,p}(U)$, and set $v = F(u)$. Prove that

$$v \in W^{1,p}(U) \quad \text{with} \quad v_{x_i} = F'(u)u_{x_i} \quad \text{for} \quad 1 \le i \le n.$$

$\boxed{3}$ Suppose $c(x) \ge 0$ is a smooth function on $\overline{U}$. Show existence of a weak solution $u \in H_0^1(U)$ of the Dirichlet problem

(2) $$\begin{cases} Lu + cu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

for arbitrary $f \in L^2(U)$. (First define what you mean by a weak solution of this problem!)

4 The purpose of this exercise is to prove a version of Poincaré's inequality for an unbounded set $U \subset \mathbb{R}^n$, contained between two parallel hyperplanes. By rotation, we may assume without loss of generality that the hyperplanes in question are given by $x_1 = a$ and $x_1 = b$, where $a < b$. Thus, we assume

$$U \subset \{x \in \mathbb{R}^n : a \le x_1 \le b\}.$$

Prove there exists $C = C(U)$ such that

$$\|u\|_{L^2(U)} \le C\,\|Du\|_{L^2(U)} \qquad \text{for all } u \in H_0^1(U).$$

(*Hint:* If $u \in C_c^\infty(U)$, write $\int u^2\,dx = \int 1 \cdot u^2\,dx$ and integrate by parts using $1 = \frac{\partial}{\partial x_1}(x_1)$.)

5 Define

$$\lambda_1 = \lambda_1(U) = \inf_{u \in C_c^\infty(U), u \neq 0} \frac{\int_U |Du|^2\,dx}{\int_U u^2\,dx}.$$

Prove:

a) $\lambda_1 > 0$.

b) For all $f \in L^2(U)$ and for all constants $\mu > -\lambda_1$, there exists a weak solution $u \in H_0^1(U)$ of

$$\text{(3)} \qquad \begin{cases} -\Delta u + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

6 Consider the Neumann problem for the Poisson equation:

$$\text{(4)} \qquad \begin{cases} -\Delta u = f & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial U, \end{cases}$$

where $f \in L^2(U)$ is given. A function $u \in H^1(U)$ is said to be a *weak solution* of (4) if (think about why this definition makes sense!)

$$B[u,v] = \int f v\,dx \qquad \text{for all } v \in H^1(U),$$

where

$$B[u,v] = \int_U \nabla u \cdot \nabla v\,dx.$$

a) Show that if a weak solution exists, then necessarily,

$$\text{(5)} \qquad \int_U f\,dx = 0.$$

Our next aim is to prove that the condition (5) is also *sufficient* for the existence of a weak solution. We introduce the notation

$$\overline{u} = \int_U u\,dx$$

for the (non-normalized) mean of a function $u \in L^1(U)$. (Observe that $L^2(U) \subset L^1(U)$ since U is bounded!)

You will probably need to make use of the following version of the Poincaré inequality (see §5.8.1 in Evans):

$$\text{(6)} \qquad \|u - \overline{u}\|_{L^2(U)} \le C\,\|Du\|_{L^2(U)} \qquad \text{for all } u \in H^1(U),$$

where $C = C(U)$.

b) Define the vector subspace
$$H = \{u \in H^1(U) : \overline{u} = 0\}.$$
Prove that $B[u,v]$ is an inner product on H.

c) Prove that H equipped with the inner product $B[u,v]$ is a Hilbert space.

d) Prove existence of a weak solution of (4), for any $f \in L^2(U)$ satisfying (5).

$\boxed{7}$ We consider the nonlinear Dirichlet problem

$$\begin{cases} -\Delta u + c(u) = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{7}$$

where $f \in L^2(U)$ is given, and $c : \mathbb{R} \to \mathbb{R}$ is a given smooth function with bounded derivative:
$$|c'| \le M.$$

a) Give a reasonable definition of a weak solution $u \in H_0^1(U)$ of (7).

b) Suppose $u : U \to \mathbb{R}$. Prove that
$$\left|D_k^h[c(u)](x)\right| \le M \left|D_k^h u(x)\right| \qquad \text{for } x \in U,\ 0 < |h| < \operatorname{dist}(x, \partial U).$$

c) Suppose $u \in H_0^1(U)$ is a weak solution of (7). Moreover, for simplicity assume u is compactly supported in U. Show that
$$u \in H^2(U).$$
(Proceed as in the proof of interior regularity for $Lu = f$; you don't need the cutoff function, since we are assuming compact support).

$\boxed{8}$ Solve the following Cauchy problems, and state where the solution is defined:

a) $xu_x + u_y = y$ with $u(x,0) = x^2$.

b) $xu_x + yu_y + u_z = u$ with $u(x,y,0) = h(x,y)$, where h is given.

c) $u_x + u^2 u_y = 1$ with $u(x,0) = 1$.

d) $u_x + u_y = u^2$ with $u(0,y) = e^{-y^2}$.

$\boxed{9}$ Solve Burgers' equation (and draw a picture showing the characteristics and shock curves)
$$u_t + uu_x = 0$$
subject to the initial conditions:

a)
$$u(x,0) = \begin{cases} 1 & x \le 0, \\ 1 - x & 0 \le x \le 1, \\ 0 & x \ge 1. \end{cases}$$

b)
$$u(x,0) = \begin{cases} 2 & x \le 0, \\ 2 - x & 0 \le x \le 1, \\ 0 & x \ge 1. \end{cases}$$

10 Consider the following boundary/inital value problem for the wave equation with dissipation:

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } U \times [0,T], \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g, \quad u_t = h & \text{on } U \times \{t=0\}. \end{cases}$$

Here $\alpha > 0$ is a dissipation constant.

a) Use the energy method to prove uniqueness of smooth solutions.

b) Prove finite speed of propagation, i.e., prove the analogue of Theorem 6 in §2.4 of Evans, for $u_{tt} - \Delta u + \alpha u_t = 0$.

11 Let $G(x,y)$ denote the Green's function on the domain U (for $-\Delta$).

a) Use the weak maximum principle for harmonic functions to prove that $G(x,y) \geq 0$ for $x,y \in U$, $x \neq y$.

b) Use the strong maximum principle for harmonic functions to prove that $G(x,y) > 0$ for $x,y \in U$, $x \neq y$.

12 **a)** Suppose $u : U \to \mathbb{R}$ is smooth. Prove that if u has a local maximum at a point $x \in U$, then

$$Du(x) = 0 \qquad \text{and} \qquad \sum_{i,j=1}^{n} u_{x_i x_j}(x)\xi_i \xi_j \leq 0 \quad \text{for all } \xi \in \mathbb{R}^n.$$

b) Now suppose $u \in C^\infty(\overline{U})$ and

$$\Delta u = 0 \qquad \text{in } U.$$

Use part (a) to prove the weak maximum principle:

$$u(x) \leq M := \max_{\partial U} u \quad \text{for all } x \in U.$$

(*Hint:* Define $u_\varepsilon(x) = u(x) + \varepsilon |x|^2$ for $\varepsilon > 0$, and use part (a) to prove that u_ε can have no interior maximum.)

13 Prove the weak maximum principle for the heat equation, i.e., part (i) of Theorem 4 in §2.3.3 in Evans, using the same ideas as in the previous problem.

14 Prove the strong maximum principle (i.e., the solution cannot attain its maximum at an interior point) for the PDE

$$u_{xx} + 5u_{yy} + \sin(u_x) = 2 \quad \text{in } U \subset \mathbb{R}^2.$$

(Use part (a) of Problem 12.)

15 Let $r = \sqrt{x^2 + y^2 + z^2}$ and assume C is a constant. Consider the Neumann problem

$$\begin{cases} \Delta u = C & \text{when } r < 1, \\ \dfrac{\partial u}{\partial r} = 2 & \text{when } r = 1. \end{cases}$$

Prove that no smooth solution exists if $C \neq 6$. Also, construct a solution if $C = 6$. (*Hint:* Symmetry.)