Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2006

Solutions for exam questions

$\boxed{1}$ **a)** Let us write

$$U_T = (a,b) \times (0,T),$$

so the closure is

$$\overline{U}_T = [a,b] \times [0,T]$$

As usual, we denote by Γ_T the parabolic boundary, consisting of the bottom and vertical sides of the boundary of $[a,b] \times [0,T]$. Thus,

$$\Gamma_T = \{(x,t) \in \overline{U}_T : x = a \text{ or } x = b \text{ or } t = 0\}.$$

Now suppose $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ is a solution of the heat equation $u_t = u_{xx}$ in U_T. Define

$$M = \max_{\overline{U}_T} u.$$

Then the weak maximum principle says that *the maximum M is attained at some point on the parabolic boundary* Γ_T. In symbols,

$$\max_{\Gamma_T} u = M.$$

b) It is trivial to verify that $u = -2xt - x^2$ satisfies $u_t = xu_{xx}$. To check whether u violates the weak maximum principle, we need to locate the maximum of u in the rectangle $R = [-2,2] \times [0,1]$. We first check interior points, so we solve

$$u_t = u_x = 0,$$

which has only one solution, namely $x = t = 0$, where $u = 0$. Now we check the boundary. For $t = 0$ we have $u \le 0$. For $x = -2$ and $0 \le t \le 1$, we have $u = 4(t-1) \le 0$. For $x = 2$ and $0 \le t \le 1$, we have $u = -4(t+1) < 0$. Finally, for $t = 1$ we have $u = -2x - x^2$, which has maximum $u = 1$ at $x = -1$. So clearly, the maximum of u over R is $u = 1$, and it occurs *only* at the top part of the boundary, at the point $(x,t) = (-1,1)$, hence the weak maximum principle is violated.

$\boxed{2}$ We have two jumps in the initial data. The first jump (at $x = 0$) gives a rarefaction solution, the other jump (at $x = 1$) gives a shock solution, which by Rankine-Hugoniot (with $F(u) = u^2/2$) moves with speed $1/2$ to the right. Combining the rarefaction and shock solutions, we get

$$u(x,t) = \begin{cases} 0 & \text{for } x \le 0, \\ \dfrac{x}{t} & \text{for } 0 \le x \le t, \\ 1 & \text{for } t \le x \le 1 + \dfrac{t}{2}, \\ 0 & \text{for } x \ge 1 + \dfrac{t}{2}. \end{cases} \tag{1}$$

Note, however, that this is valid only until the time t when the rarefaction and the shock meet:

$$t = 1 + \frac{t}{2} \qquad \text{i.e.,} \qquad t = 2.$$

At this time we have

$$u(x,2) = \begin{cases} 0 & \text{for } x \le 0, \\ \dfrac{x}{2} & \text{for } 0 \le x \le 2, \\ 0 & \text{for } x \ge 2. \end{cases}$$

So now we solve the intial value problem starting from $t = 2$ with these initial data. Then we get a shock emanating from $x = 2$, moving to the right along a path

$$x = \xi(t),$$

and to the left of the shock we have the continuation of the rarefaction wave,

$$u = u_l = \frac{x}{t},$$

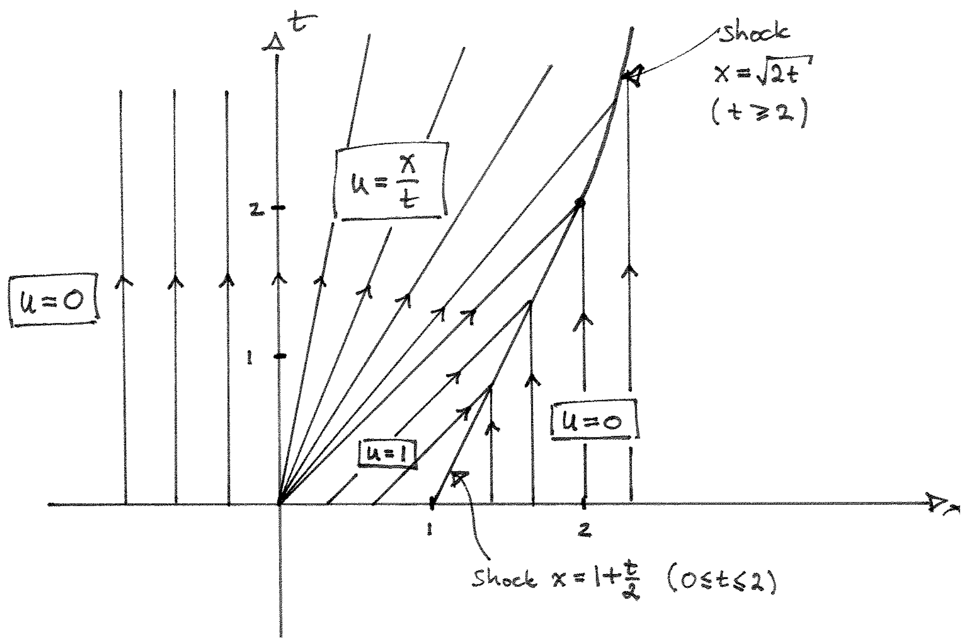
while $u = u_r = 0$ to the right. Thus, by Rankine-Hugoniot,

$$\frac{d\xi}{dt} = \frac{1}{2}u_l = \frac{\xi}{2t},$$

which we separate and solve to get $\xi = \sqrt{2t}$, since the initial condition is $\xi(2) = 2$. We conclude that for $t \ge 2$,

$$u(x,t) = \begin{cases} 0 & \text{for } x \le 0, \\ \dfrac{x}{t} & \text{for } 0 \le x \le \sqrt{2t}, \\ 0 & \text{for } x \ge \sqrt{2t}, \end{cases} \tag{2}$$

which together with (1) for $0 \le t \le 2$ provides the complete solution.



$\boxed{3}$ Suppose $u, v : \overline{U} \times [0,T]$ are two smooth solution of the initial/boundary value problem. Then $w = u - v$ is a solution of the same problem, but with $f, g, h = 0$. Now define

$$E(t) = \frac{1}{2}\int_U w_t^2 + |\nabla w|^2 \, dx,$$

where the integrand is understood to be evaluated at (x,t). Then

$$\begin{aligned} E'(t) &= \int_U w_t w_{tt} + \nabla w \cdot \nabla w_t \, dx \\ &= \int_U w_t (w_{tt} - \Delta w)\, dx + \int_{\partial U} \frac{\partial w}{\partial \nu} w_t \, dx \\ &= 0 \end{aligned}$$

where we used integration by parts and the fact that

$$\frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial U \times [0,T].$$

Thus, $0 \le E(t) \le E(0)$ for $0 \le t \le T$, but $E(0) = 0$, since $f = g = 0$. Thus $Dw = 0$ in $U \times [0,T]$, hence $w = \text{const.}$ in $U \times [0,T]$, but this constant must be zero, since we have zero initial data.

$\boxed{4}$ For simplicity let's switch to (x_1, x_2, x_3) instead of (x, y, z). By Kirchhoff's formula,

$$u(x,t) = D_t\left(\frac{t}{4\pi}\int_{|y|=1} f(x+ty)\, dS(y)\right),$$

where

$$f(x) = x_1^2 + x_2^2.$$

Note that

$$f(x+ty) = x_1^2 + x_2^2 + 2t(x_1y_1 + x_2y_2) + t^2(y_1^2 + y_2^2).$$

Now we use:

$$\begin{aligned} \int_{|y|=1} y_i \, dS(y) &= 0, \\ \int_{|y|=1} 1\, dS(y) &= 4\pi, \end{aligned}$$

$$\int_{|y|=1} y_i^2 \, dS(y) = \frac{1}{3}\int_{|y|=1} \sum_{j=1}^{3} y_j^2 \, dS(y) = \frac{1}{3}\int_{|y|=1} 1\, dS(y) = \frac{4\pi}{3}.$$

It follows that

$$\frac{t}{4\pi}\int_{|y|=1} f(x+ty)\, dS(y) = t(x_1^2 + x_2^2) + \frac{2t^3}{3},$$

hence

$$u = \underline{\underline{x_1^2 + x_2^2 + 2t^2}}.$$

$\boxed{5}$ **a)** A weak solution of the given Dirichlet problem is a function $u \in H_0^1(U)$ such that

(3) $$\underbrace{\int_U \nabla u \cdot \nabla v \, dx}_{=B[u,v]} = \int_U f v \, dx \qquad \text{for all } v \in H_0^1(U).$$

By the Poincaré inequality, which reads (here C depends only on U)

$$\|u\|_{L^2(U)} \le C\,\|Du\|_{L^2(U)} \qquad \text{for all } u \in H_0^1(U),$$

we have that $B[u,u] = 0 \implies u = 0$, so $B[u,v]$ is an inner product on $H_0^1(U)$. Moreover, the norm $\|u\| = \sqrt{B[u,u]} = \|Du\|_{L^2(U)}$ is equivalent to the standard norm on $H_0^1(U)$, again by Poincaré's inequality. Indeed,

$$\|u\| \le \|u\|_{H_0^1(U)} = \|u\|_{L^2(U)} + \|Du\|_{L^2(U)} \le C\,\|u\|.$$

Finally, existence of a weak solution follows by the Riesz representation theorem applied to the Hilbert space $H_0^1(U)$, since $v \mapsto \int_U f\,v\,dx$ is a bounded linear functional on $H_0^1(U)$:

$$\left|\int_U f\,v\,dx\right| \le \|f\|_{L^2(U)}\,\|u\|_{L^2(U)} \le C\,\|f\|_{L^2(U)}\,\|u\|,$$

for all $v \in H_0^1(U)$.

b) Suppose $u \in H_0^1(U)$ is a weak solution and $\operatorname{supp} u \subset\subset U$. Let $k \in \{1,\dots,n\}$. We take

$$v = D_k^{-h} D_k^h u$$

in (3). Then

$$\underbrace{\int_U \nabla u \cdot \nabla D_k^{-h} D_k^h u\,dx}_{=A} = \underbrace{\int_U f D_k^{-h} D_k^h u\,dx}_{=B}.$$

By "integration by parts" for difference quotients (valid by the support assumption on u),

$$A = -\int_U \nabla D_k^h u \cdot \nabla D_k^h u\,dx = -\left\|\nabla D_k^h u\right\|_{L^2(U)}^2 = -\left\|D_k^h \nabla u\right\|_{L^2(U)}^2.$$

By Cauchy's inequality with ε, we have

$$|B| \le \frac{1}{4\varepsilon}\,\|f\|_{L^2(U)}^2 + \varepsilon\left\|D_k^{-h} D_k^h u\right\|_{L^2(U)}^2 \le \frac{1}{4\varepsilon}\,\|f\|_{L^2(U)}^2 + \varepsilon\left\|\nabla D_k^h u\right\|_{L^2(U)}^2,$$

where we used Theorem 3(i) in §5.8.2 of Evans (which holds with constant $C = 1$, actually) to get the last inequality.
We conclude:

$$\left\|D_k^h \nabla u\right\|_{L^2(U)}^2 = -A = -B \le \frac{1}{4\varepsilon}\,\|f\|_{L^2(U)}^2 + \varepsilon\left\|D_k^h \nabla u\right\|_{L^2(U)}^2,$$

hence, choosing $\varepsilon = 1/2$,

$$\left\|D_k^h \nabla u\right\|_{L^2(U)}^2 \le \|f\|_{L^2(U)}^2.$$

Since this holds for all small enough $h \ne 0$, it follows by Theorem 3(ii) in §5.8.2 that the second order weak partial derivatives exist, and

$$\|u_{x_j x_k}\|_{L^2(U)}^2 \le \|f\|_{L^2(U)}^2 \qquad (j,k = 1,\dots,n).$$

Therefore $u \in H^2(U)$.