Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2007

Supplementary notes 1

Our aim here is to establish *domain of dependence/finite speed of propagation* for the wave equation with lower-order linear terms. The material here supersedes the discussion in Section 3.4c of McOwen.

So we consider

$$\underbrace{u_{tt} - c^2\Delta u}_{\text{principal part}} + \underbrace{b(x,t)\cdot Du + a(x,t)u}_{\text{lower-order terms}} = 0 \qquad (u = u(x,t),\ x \in \mathbb{R}^n,\ t \in \mathbb{R}), \tag{1}$$

where we use the notation

$$Du = (c\nabla, u_t)$$

and

$$b = (b_1, \cdots, b_{n+1}),$$

so that

$$b \cdot Du = (b_1 c)u_{x_1} + \cdots + (b_n c)u_{x_n} + b_{n+1}u_t.$$

Moreover, we assume that $a(x,t)$ and $b_j(x,t)$ are continuous and bounded. Thus, there exists $M < \infty$ such that

$$1 + |a(x,t)| + |b(x,t)| \le M \qquad \text{for all } x \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{2}$$

Now fix a point $P = (x_0, t_0)$ with $t_0 > 0$. Let Λ_P denote the domain of dependence of the point P, as for the standard wave equation (without lower-order terms). Specifically,

$$\Lambda_P = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \le t \le t_0,\ |x - x_0| \le c(t_0 - t)\right\}.$$

We shall show that this is also the domain of dependence for the wave equation *with* lower order terms. (Therefore, it is the principal part which determines the domain of dependence, not the lower-order terms.) The time-slices of the cone Λ_P are defined by

$$B_t = \left\{x \in \mathbb{R}^n : (x,t) \in \Lambda_P\right\} = \left\{x \in \mathbb{R}^n : |x - x_0| \le c(t_0 - t)\right\} \qquad (0 \le t \le t_0).$$

So B_t is just the closed ball in $\mathbb{R}^n$ of radius $c(t_0 - t)$ and centered at 0.

We shall prove the following theorem (this generalizes Theorem 2 in Section 3.3 of McOwen).

Theorem 1. *With notation as above, suppose $u \in C^2(\Lambda_P)$ satisfies (1) in Λ_P, with zero initial data in the base:*

$$u(x,0) = u_t(x,0) = 0 \qquad \textit{for } x \in B_0.$$

Then $u = 0$ in Λ_P.

Before proving this, let us note the following corollary:

Corollary (Uniqueness). *Uniqueness holds for solutions $u \in C^2(\Lambda_P)$ of the Cauchy problem*

$$\begin{cases} u_{tt} - c^2\Delta u + b(x,t)\cdot Du + a(x,t)u = f(x,t), & (x,t) \in \Lambda_P, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in B_0. \end{cases} \tag{3}$$

To prove this corollary, assume that $u, v \in C^2(\Lambda_P)$ both solve (3). Then $w = u - v$ solves (3) with $f = 0$ and $g = h = 0$, so Theorem 1 implies that $w = 0$, and hence $u = v$, in Λ_P.

This corollary shows that Λ_P is the domain of dependence of P. Indeed, suppose now that u is a C^2 solution of (3), but on the whole space $\mathbb{R}^n \times \mathbb{R}$, not just on Λ_P. Then the corollary shows that $u(P)$ can only depend on the values of $f(x,t)$ for $(x,t) \in \Lambda_P$ and on $g(x)$ and $h(x)$ for x in the base B_0 of Λ_P.

We now prove Theorem 1.

Step 1. We define

$$\mathscr{E}(t) = \frac{1}{2}\int_{B_t} \underbrace{u_t^2 + c^2|\nabla u|^2 + u^2}_{\text{call this } e = e(x,t)}\,dx \qquad (0 \le t \le t_0).$$

Note that $\mathscr{E}(0) = 0$. Our plan is to show that $\mathscr{E}(t) = 0$ for all $0 \le t \le t_0$. If we are able to do this, then it follows that $e(x,t) = 0$ for all $(x,t) \in \Lambda_P$ (here we rely on the fact that $e(x,t) \ge 0$ by definition), so in particular $u(x,t) = 0$, and then we are done.

Step 2. We calculate $\mathscr{E}'(t)$. To this end we rewrite the integral defining $\mathscr{E}(t)$, by passing to polar coordinates $x = x_0 + ry$, where $r = |x - x_0|$ and y is on the unit sphere in $\mathbb{R}^n$. Thus, $0 \le r \le c(t_0 - t)$, and $y \in \mathbb{R}^n$, $|y| = 1$. Let $dS(y)$ be the surface area element on the unit sphere. Thus,[1]

$$\mathscr{E}(t) = \frac{1}{2}\int_{B_t} e(x,t)\,dx = \frac{1}{2}\int_0^{c(t_0-t)} \left(\int_{|y|=1} e(x_0 + ry, t)\,dS(y)\right) r^{n-1}\,dr.$$

The advantage now is that it is easy to take the derivative; by the usual rule for differentiating an integral with respect to a parameter,[2]

$$\mathscr{E}'(t) = -\frac{c}{2}(c[t_0 - t])^{n-1}\int_{|y|=1} e\big(x_0 + c[t_0 - t]y, t\big)\,dS(y) + \frac{1}{2}\int_0^{c(t_0-t)} \left(\int_{|y|=1} e_t(x_0 + ry, t)\,dS(y)\right) r^{n-1}\,dr.$$

Having differentiated, it is more convenient to switch back to the original variables. For the first integral, we note that $x = x_0 + c[t_0 - t]y$ is a point on the boundary ∂B_t of B_t, and the relation between the surface area elements is $dS(x) = (c[t_0 - t])^{n-1}\,dS(y)$ (the areas in question are $(n-1)$-dimensional). Thus,

$$\begin{aligned}
\mathscr{E}'(t) &= -\frac{c}{2}\int_{\partial B_t} e(x,t)\,dS(x) + \frac{1}{2}\int_{B_t} e_t(x,t)\,dx \\
&= -\frac{c}{2}\int_{\partial B_t} u_t^2 + c^2|\nabla u|^2 + u^2\,dS(x) + \int_{B_t} u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t + u u_t\,dx \\
&= -\frac{c}{2}\int_{\partial B_t} u_t^2 + c^2|\nabla u|^2 + u^2\,dS(x) + \int_{B_t} u_t\left(u_{tt} - c^2\Delta u + u\right)dx + \int_{\partial B_t} c^2 u_t(\nabla u \cdot \nu)\,dS(x) \\
&= \int_{\partial B_t} c\left[u_t c(\nabla u \cdot \nu) - \frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2 + u^2\right)\right] dS(x) + \frac{1}{2}\int_{B_t} u_t\big(-b \cdot Du + (1-a)u\big)\,dx \\
&= I + J,
\end{aligned}$$

where in the third step we used the general integration by parts formula,[3] and in the fourth step we used the equation (1). Note that ν is the exterior unit normal on the sphere ∂B_t.

[1] If you prefer, just take $n = 3$. Then you can use spherical coordinates $y = (\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta)$, where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. Then $dS(y) = \sin\phi\,d\phi\,d\theta$, hence

$$\mathscr{E}(t) = \frac{1}{2}\int_{B_t} e(x,t)\,dx = \frac{1}{2}\int_0^{c(t_0-t)} \left(\int_0^{2\pi}\int_0^{\pi} e\big(x_0 + r(\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta), t\big)\sin\phi\,d\phi\,d\theta\right) r^2\,dr.$$

[2] The rule we need is $\frac{d}{dt}\left(\int_0^{\alpha(t)} F(s,t)\,ds\right) = F(\alpha(t),t)\cdot\alpha'(t) + \int_0^{\alpha(t)} F_t(s,t)\,ds$. This is valid, e.g., if $F(s,t)$ and $F_t(s,t)$ are both continuous functions, which is the case in the present application.

[3] This formula reads

$$\int_\Omega f_{x_i} g\,dx = -\int_\Omega f g_{x_i}\,dx + \int_{\partial\Omega} f g \nu_i\,dS,$$

for $f, g \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $f_{x_i}, g_{x_i} \in L^1(\Omega)$. Here Ω is a bounded domain with smooth boundary, and ν is the exterior unit normal on $\partial\Omega$.

Step 3. We estimate $\mathscr{E}'(t)$. From Step 2 we have $\mathscr{E}'(t) = I + J$. First we estimate

$$I = \int_{\partial B_t} c \left[u_t c(\nabla u \cdot \nu) - \frac{1}{2}\left(u_t^2 + c^2 |\nabla u|^2 + u^2\right)\right] dS(x) \le -\frac{c}{2}\int_{\partial B_t} u^2 \, dS(x) \le 0, \tag{4}$$

where we used the fact that for any real numbers q and r, we have

$$qr \le \frac{1}{2}(q^2 + r^2), \tag{5}$$

and we used the Cauchy-Schwarz inequality, which states that if x and y are vectors in $\mathbb{R}^n$, then

$$\left| x \cdot y \right| \le |x| \left| y \right|. \tag{6}$$

Together these inequalities imply (recall also that $|\nu| = 1$)

$$u_t c(\nabla u \cdot \nu) \le \frac{1}{2}\left(u_t^2 + c^2 |\nabla u \cdot \nu|^2\right) \le \frac{1}{2}\left(u_t^2 + c^2 |\nabla u|^2\right),$$

which gives (4).

Next, we estimate

$$\begin{aligned}
J &= \frac{1}{2}\int_{B_t} u_t\big(-b \cdot Du + (1-a)u\big)\, dx \\
&\le \frac{1}{2}\int_{B_t} u_t\big(|b|\,|Du| + [1 + |a|]\,|u|\big)\, dx \\
&\le \frac{M}{2}\int_{B_t} u_t\big(|Du| + |u|\big)\, dx && \text{by (2)} \\
&\le \frac{M}{2}\int_{B_t} \left[\frac{1}{2}\left(u_t^2 + |Du|^2\right) + \frac{1}{2}\left(u_t^2 + u^2\right)\right] dx && \text{by (5)} \\
&\le M \int_{B_t} \left[u_t^2 + c^2 |\nabla u|^2 + u^2\right] dx && \text{since } |Du|^2 = u_t^2 + c^2 |\nabla u|^2
\end{aligned} \tag{7}$$

Step 4. From (4) and (7) we conclude that

$$\mathscr{E}'(t) \le M\mathscr{E}(t) \qquad (0 \le t \le t_0).$$

To solve this *differential inequality*, as it is called, we should think about how we would solve the corresponding differential *equation*, if we had equality instead of inequality. Thus, it is natural to try to multiply both sides by the integrating factor $\exp(-Mt)$. This gives

$$\frac{d}{dt}\left(e^{-Mt}\mathscr{E}(t)\right) \le 0,$$

so the function $e^{-Mt}\mathscr{E}(t)$ is nonincreasing. Therefore, $e^{-Mt}\mathscr{E}(t) \le \mathscr{E}(0)$, i.e.,

$$\mathscr{E}(t) \le e^{Mt}\mathscr{E}(0) \qquad (0 \le t \le t_0).$$

But in our case, $\mathscr{E}(0) = 0$, so we finally conclude that

$$\mathscr{E}(t) = 0 \qquad (0 \le t \le t_0),$$

which completes the proof of Theorem 1.