LECTURE NOTES
TMA4305 PARTIAL DIFFERENTIAL EQUATIONS, SPRING 2007

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1. BASIC FUNCTIONAL ANALYSIS

Let X be a vector space over $\mathbb{R}$.

1.1. **Normed spaces.**

Definition 1. A *norm* on X is a map $\|\cdot\| : X \to [0,\infty)$, $x \mapsto \|x\|$, such that

(i) $\|x\| = 0 \implies x = 0$,

(ii) $\|cx\| = |c|\,\|x\|$,

(iii) $\|x + y\| \le \|x\| + \|y\|$,

for $c \in \mathbb{R}$ and $x, y \in X$. The pair $(X, \|\cdot\|)$ is then called a *normed vector space*, or just *normed space* for short.

Observe that if X is a normed space with norm $\|\cdot\|$, then $d(x, y) = \|x - y\|$ is a distance function, or *metric*, on X. Thus, every normed space is a metric space.

Definition 2. A *Banach space* is a normed space X which is complete as a metric space, i.e., every Cauchy sequence in X has a limit in X.

Recall that a *Cauchy sequence* in X is a sequence $\{x_j\} \subset X$ such that for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\|x_j - x_k\| \le \varepsilon \qquad \text{for all } j, k \ge N.$$

This sequence has a limit in X if there exists $x \in X$ such that $\lim_{j\to\infty} x_j = x$ in X, i.e.,

$$\lim_{j\to\infty} \|x_j - x\| = 0.$$

1.2. **Inner product spaces.**

Definition 3. An *inner product* on X is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that

(i) $\langle x, x, \ge \rangle\, 0$ for all $x \in X$, and $\langle x, x, = \rangle\, 0$ if and only if $x = 0$,

(ii) $\langle ax + by, z \rangle = a\langle x, z\rangle + b\langle y, z\rangle$,

(iii) $\langle x, y \rangle = \langle y, x \rangle$,

for $a, b \in \mathbb{R}$ and $x, y, z \in X$. The pair $(X, \langle \cdot, \cdot \rangle)$ is then called an *inner product space*.

If X is an inner product space with inner product $\langle\cdot,\cdot\rangle$, then

$$\|x\| = \sqrt{\langle x, x\rangle}$$

is a norm on X, which we refer to as the norm *associated to the inner product* $\langle\cdot,\cdot\rangle$. The triangle inequality $\|x+y\| \le \|x\| + \|y\|$ is satisfied on account of the *Cauchy-Schwarz inequality*:

$$\left|\left\langle x, y\right\rangle\right| \le \|x\| \left\|y\right\|,$$

which holds in every inner product space.

Definition 4. A *Hilbert space* is an inner product space that is a Banach space with respect to the associated norm.

The usual notation for a Hilbert space is H (instead of X).

1.3. **Linear maps and linear functionals.** A linear map $T : X \to Y$ between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is *bounded* if there exists $C \ge 0$ such that

$$\|Tx\|_Y \le C\|x\|_X \qquad \text{for all } x \in X.$$

The best constant C in this estimate is called the *operator norm* and denoted $\|T\|$. Thus,

$$\|T\| = \sup_{x\in X, x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

Exercise 1. Prove that the operator norm is indeed a norm on the space $L(X,Y)$ of bounded linear maps from X to Y. Moreover, prove that if Y is a Banach space, then so is $L(X,Y)$, with the operator norm.

The following theorem is often useful:

Theorem 1. *Suppose X is a normed space, Y is a Banach space, and $D \subset X$ is a dense subspace. Then any bounded linear map $S : D \to Y$ has a unique extension to a bounded linear map from X to Y.*

Exercise 2. Prove this theorem.

A *linear functional* on X is a linear map $F : X \to \mathbb{R}$. The *dual space* of X is the space of bounded linear functionals on X and is denoted X^*. We equip this space with the operator norm. Then by Exercise 1, X^* is a Banach space.

If H is a Hilbert space, then for any $y \in H$, the map $F : H \to \mathbb{R}$ defined by $F(x) = \left\langle x, y\right\rangle$ is a bounded linear functional on H, by the Cauchy-Schwarz inequality. In fact, *every* bounded linear functional on H is obtained in this way. This is the content of the following famous theorem, which we shall need for our applications later on.

Riesz Representation Theorem. *Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Suppose $F \in H^*$. Then there exists a unique $y \in H$ such that*

$$F(x) = \left\langle x, y\right\rangle \qquad \textit{for all } x \in H.$$

We skip the proof.

1.4. **The completion of a normed space.** A linear map $T : X \to Y$ is an *isometry* if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. Then T is of course injective, so we may view X as a subspace Y, i.e., we may identify X with its image $T(X) \subset Y$. We then say that X is isometrically embedded in Y.

The following theorem is of fundamental importance:

Theorem 2. *Every normed space X embeds isometrically as a dense subspace of a Banach space $\overline{X}$. To be precise, there exist a Banach space $\overline{X}$ and a linear isometry $\iota : X \to \overline{X}$ such that $\iota(X)$ is dense in $\overline{X}$.*

The space $\overline{X}$ is called the *completion* of X, and is uniquely determined, up to isometry: If we have another Banach space $\tilde{X}$ and a linear isometry $\tilde{\iota} : X \to \tilde{X}$ such that $\tilde{\iota}(X)$ is dense in $\tilde{X}$, then $\tilde{\iota} \circ \iota^{-1} : \iota(X) \to \tilde{X}$ is a linear isometry defined on a dense subspace of $\overline{X}$, so by Theorem 1 it can be extended uniquely to a linear isometry $T : \overline{X} \to \tilde{X}$. To prove that this map is onto, suppose $y \in \tilde{X}$. Since $T(X)$ contains $\tilde{\iota}(X)$, which is dense in $\tilde{X}$, there exists a sequence $\{x_j\}$ in X such that $Tx_j \to y$; but then, since T is an isometry, x_j must be Cauchy, hence converges to some $x \in X$, and it follows that $y = Tx$.

Exercise 3. Prove Theorem 2, by following these steps:

(a) Consider the vector space of all Cauchy sequences $\{x_j\}$ in X; we shall consider two such sequences $\{x_j\}$ and $\{y_j\}$ to be *identical* if $\lim_{j\to\infty} \|x_j - y_j\| = 0$; we call the resulting vector space $\overline{X}$. Now define

$$\|\{x_j\}\| = \lim_{j\to\infty} \|x_j\|.$$

Show that this is a well-defined norm on $\overline{X}$.

(b) Define $\iota : X \to \overline{X}$ by

$$\iota(x) = \{x, x, \dots\}.$$

Show that ι is a linear isometry and that $\iota(X)$ is dense in $\overline{X}$.

(c) Prove the following general Lemma: Suppose Y is a normed space and $D \subset Y$ is a dense subspace. If every Cauchy sequence $\{y_j\} \subset D$ has a limit in Y, then Y is complete.

(d) Prove that $\overline{X}$ is complete.

Remark 1. Suppose Y is a Banach space containing X as a subspace, and that the norms on X and Y, which we denote $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, satisfy

$$\|x\|_Y \le C\|x\|_X \qquad \text{for all } x \in X. \tag{1}$$

(In other words, the inclusion map from X into Y is bounded.) Then the completion $\overline{X}$ of X can be realized as a subspace of Y. In fact, this follows from the construction in the above exercise; performing this construction for both the spaces X and Y, we obtain spaces $\overline{X}$ and $\overline{Y}$ of Cauchy sequences (again identifying two sequences whose difference converges to zero), and linear isometries $\iota_X : X \to \overline{X}$ and $\iota_Y : Y \to \overline{Y}$. But since Y is complete, the map ι_Y is in fact an isometric isomorphism. Moreover, on account of (1), every Cauchy sequence in X is also a Cauchy sequence in Y, hence we can identify $\overline{X}$ with a subspace of $\overline{Y}$ in an obvious way, and then $\iota_Y|_X = \iota_X$. Finally, we use the isomorphism ι_Y to map everything back into Y. Thus, we see that $\iota_Y^{-1}(\overline{X}) \subset Y$ can be used as the completion of X, the norm being $\|x\| = \|\iota_Y(x)\|_{\overline{X}}$ (note that this norm agrees with $\|x\|_X$ in case $x \in X$, since $\iota_Y|_X = \iota_X$).

Let us note also the following:

Theorem 3. *If X is an inner product space, then the completion of X is a Hilbert space.*

1.5. **Examples.**

1.5.1. *The space $BC(E)$.* Suppose $E \subset \mathbb{R}^n$. We denote by $BC(E)$ the space of functions $f : E \to \mathbb{R}$ which are continuous and bounded. This is a Banach space when equipped with the norm

$$\|f\|_\infty = \sup_{x \in E} |f(x)|.$$

This is called the *uniform norm*, since convergence $f_j \to f$ in this norm is nothing else than uniform convergence on E.

As a particular case, note that if $E = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded, open set (as is typical in our applications), then the boundedness is automatic, hence

$$BC(\overline{\Omega}) = C(\overline{\Omega}),$$

and the norm is a maximum:

$$\|f\|_\infty = \max_{x \in \overline{\Omega}} |f(x)|.$$

Exercise 4. Prove that $BC(E)$ is a Banach space with the uniform norm.

1.5.2. *The space $C^1(\overline{\Omega})$.* This is the space of functions $f \in C^1(\Omega)$ such that f and all the first order partials $\partial_j f$ extend continuously to $\overline{\Omega}$. This is a Banach space if we use the norm

$$\|f\|_{1,\infty} = \sup_{x \in \Omega} |f(x)| + \sup_{x \in \Omega} |\nabla f(x)|.$$

Alternatively, we can use the norm

$$|f|_{1,\infty} = \sup_{x \in \Omega} \left(|f(x)| + |\nabla f(x)| \right).$$

These two norms are equivalent: Trivially, $|f|_{1,\infty} \le \|f\|_{1,\infty}$. To prove the converse inequality, let us we write $\|f\|_{1,\infty} = a + b$, where $a = \sup_{x \in \Omega} |f(x)|$ and $b = \sup_{x \in \Omega} |\nabla f(x)|$. Let us assume $b \le a$. For any $\varepsilon > 0$, we can find $x \in \Omega$ such that $a - \varepsilon \le |f(x)|$. Then it follows that $\|f\|_{1,\infty} = a + b \le 2a \le 2|f(x)| + 2\varepsilon \le 2|f|_{1,\infty} + 2\varepsilon$. The same argument works if $a \le b$, and letting $\varepsilon \to 0$, we conclude that $(1/2)\|f\|_{1,\infty} \le |f|_{1,\infty}$.

Exercise 5. Prove that $C^1(\overline{\Omega})$ is a Banach space with either of the above norms.

1.5.3. *The space $C^k(\overline{\Omega})$.* This is the space of functions $f \in C^k(\Omega)$ such that for all $|\alpha| \le k$, the partial $\partial^\alpha f$ extends continuously to $\overline{\Omega}$. The norm here is

$$\|f\|_{k,\infty} = \sum_{|\alpha| \le k} \sup_{x \in \Omega} |\partial^\alpha f(x)|.$$

The completeness follows from that of $C^1(\overline{\Omega})$, by an induction argument.

We shall also denote

$$C^\infty(\overline{\Omega}) = \bigcap_{k=1}^{\infty} C^k(\overline{\Omega}),$$

but we do not attempt to define a norm on this space.

1.5.4. *The space* $L^p(\Omega)$. Let $1 \le p < \infty$. The space $L^p(\Omega)$ can be defined as the completion (cf. Theorem 2) of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}.$$

(This really *is* a norm on $C_0^\infty(\Omega)$; the triangle inequality is known as *Minkowski's inequality*.)

Remark 2. If we use the Lebesgue theory of integration, then $L^p(\Omega)$ can be explicitly realized as the space of measurable functions $f : \Omega \to \mathbb{R}$ such that $\|f\|_p < \infty$. (Identifying, as usual, functions which are equal except on a set of measure zero.)

Remark 3. If we view $L^1(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_1$ (instead of using the Lebesgue theory), we can define a linear functional

$$I_0 : C_0^\infty(\Omega) \to \mathbb{R}, \qquad I_0(f) = \int_\Omega f(x)\, dx.$$

This functional is bounded, since

$$|I_0(f)| \le \int_\Omega |f(x)| \, dx = \|f\|_1.$$

Therefore, by Theorem 1, I_0 has a unique extension to a bounded linear functional $I : L^1(\Omega) \to \mathbb{R}$, and it is natural to use the notation

$$I(f) = \int_\Omega f(x)\, dx.$$

The right hand side is then the Lebesgue integral. (Of course, defining the integral *abstractly* like this is easy, but it certainly does not mean that the explicit construction of the Lebesgue integral can be avoided in all situations; the main strength of this theory is in the convergence theorems!)

Remark 4. The case $p = 2$ is of particular importance, since $L^2(\Omega)$ is a Hilbert space. Indeed, the norm $\|f\|_2$ is associated to the inner product

$$\langle f, g \rangle = \int_\Omega f(x) g(x)\, dx.$$

2. SOBOLEV SPACES

The space $C^k\left(\overline{\Omega}\right)$ is not well suited if we want to apply techniques from functional analysis to the study of PDEs. In fact, this space is too restrictive: usually we are not able to prove the estimates that would be needed. Instead, a larger, less restrictive space is better suited: the Sobolev space. Then one typically splits the problem of solving the PDE into two steps:

(i) prove existence of solutions in a Sobolev space (weak solutions, possibly);

(ii) prove that the solutions are in fact more regular, for suitable regular data.

We shall focus on step (i) in the remainder of this course; (ii) is usually more difficult.

2.1. **Basic definitions.** The idea of the Sobolev spaces is this: Consider the spaces $C^k(\overline{\Omega})$, with the norm $\sum_{|\alpha|\le k} \|\partial^\alpha f\|_\infty$. If we replace the uniform norm by the L^p norm for some $1 \le p < \infty$, we get a Sobolev space. For our present purposes it will be enough to consider only the case where $k = 1$ and $p = 2$.

We shall assume throughout that Ω is a bounded domain in $\mathbb{R}^n$. (This is not at all essential for defining the Sobolev spaces, but it suffices for our purposes, and it simplifies the presentation.)

Definition 5. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm (here the subscript 1 stands for one derivative, and 2 reflects the fact that we use the L^2 norm)

$$\|f\|_{1,2} = \left(\int_\Omega f(x)^2 + |\nabla f(x)|^2 \, dx\right)^{1/2} = \left(\|f\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|\partial_j f\|_{L^2(\Omega)}^2\right)^{1/2},$$

which is the norm associated to the inner product

$$\langle f, g\rangle_1 = \int_\Omega f(x)g(x) + \nabla f(x)\cdot\nabla g(x)\, dx.$$

Definition 6. $H^1(\Omega)$ is the completion of $C^\infty(\overline{\Omega})$ with respect to the norm $\|f\|_{1,2}$.

Note that both $H_0^1(\Omega)$ and $H^1(\Omega)$ are Hilbert spaces (cf. Theorem 3). Note also that $H_0^1(\Omega) \subset H^1(\Omega) \subset L^2(\Omega)$ (cf. Remark 1). Intuitively, we can think of $H_0^1(\Omega)$ as the space of $f \in H^1(\Omega)$ such that $f = 0$ on the boundary $\partial\Omega$. This is made more precise in the following remark.

Remark 5. (Boundary values of $f \in H^1(\Omega)$.) Since we want to study the Dirichlet problems, it is important to know that it makes sense to talk about the boundary values of a function $f \in H^1(\Omega)$. In fact, let us define

$$T_0 : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega), \qquad Tf = f|_{\partial\Omega}.$$

It is not very hard to show that if $\partial\Omega$ is smooth, then

$$\|T_0 f\|_{L^2(\partial\Omega)} \le C_\Omega \|f\|_{1,2} \qquad \text{for all } f \in C^\infty(\overline{\Omega}),$$

where the L^2 norm on the left side is defined using the surface area element dS on $\partial\Omega$. Thus, T_0 extends uniquely to a bounded linear map $T : H^1(\Omega) \to L^2(\partial\Omega)$, called the *trace operator*. We can think of Tf as the boundary values $f|_{\partial\Omega}$ of f. One can now show:

Theorem 4. *If Ω has smooth boundary, then* $H_0^1(\Omega) = \{f \in H^1(\Omega) : Tf = 0\}$.

(In fact, the inclusion $\subset$ is trivial, but the converse is not!)

2.2. **Weak derivatives.** If $f \in H^1(\Omega)$, then f has "weak" (or distributional) first order partial derivatives in $L^2(\Omega)$. Indeed, by density there exists a sequence $\{f_j\} \subset C^\infty(\overline{\Omega})$ such that $\|f_j - f\|_{1,2} \to 0$ as $j \to \infty$. In particular, this means that for each $k = 1, \dots, n$, the sequence $\{\partial_k f_j\}$ is Cauchy in $L^2(\Omega)$, hence it converges to some $g_k \in L^2(\Omega)$. So now we have:

$$f_j \to f, \qquad \partial_k f_j \to g_k \qquad \text{in } L^2(\Omega), \text{ as } j \to \infty.$$

We claim that g_k is the weak (or distributional) derivative of f with respect to x_k. To see this, pick any test function $\phi \in C_0^\infty(\Omega)$. Then by integration by parts,

$$\int_\Omega f_j(x)\partial_k v(x)\, dx = -\int_\Omega \partial_k f_j(x) v(x)\, dx,$$

or, equivalently, denoting the L^2 inner product by $\langle \cdot, \cdot \rangle$,

$$\langle f_j, \partial_k v \rangle = -\langle \partial_k f_j, v \rangle.$$

Letting $j \to \infty$, we then get (using the fact that $\langle \cdot, v \rangle$ and $\langle \cdot, \partial_k v \rangle$ are bounded linear functionals on $L^2(\Omega)$!)

$$\langle f, \partial_k v \rangle = -\langle \partial_k f, v \rangle,$$

or, equivalently,

$$\int_\Omega f(x) \partial_k v(x)\, dx = -\int_\Omega \partial_k f(x) v(x)\, dx.$$

This proves that $\partial_k f = g_k$ in the weak sense (the sense of distributions), and we then conclude: If $f \in H^1(\Omega)$, then $f, \partial_1 f, \dots, \partial_n f$ all belong to $L^2(\Omega)$.

Remark 6. Another way to express the last statement is that $H^1(\Omega) \subset W^{1,2}(\Omega)$, where (this is a standard definition)

$$W^{1,2}(\Omega) = \left\{ f \in L^2(\Omega) : \partial_k f \in L^2(\Omega),\ k = 1, \dots, n \right\},$$

the derivatives being taken in the distributional sense. We use the norm $\|f\|_{1,2}$ on this space. In fact, one can show that if Ω is a "nice" domain (for example, if the boundary is smooth), then the converse inclusion also holds, i.e., we have

$$H^1(\Omega) = W^{1,2}(\Omega).$$

The advantage of this is that it gives an explicit realization of the space $H^1(\Omega)$, instead of the abstract realization as a completion.

2.3. **Poincaré's inequality.**

Theorem 5. *If Ω is bounded, there exists $C = C(\Omega) > 0$ such that*

$$\|f\|_{L^2(\Omega)} \le C \|\nabla f\|_{L^2(\Omega)} \qquad \text{for all } f \in H_0^1(\Omega).$$

See McOwen (Theorem 6.2.1) for a short proof.

As a consequence of this inequality, we see that if Ω is bounded, then on the space $H_0^1(\Omega)$, the norms

$$\|f\|_{1,2} = \left(\int_\Omega |f(x)|^2 + |\nabla f(x)|^2\, dx \right)^{1/2}$$

and

$$|f|_{1,2} = \left(\int_\Omega |\nabla f(x)|^2\, dx \right)^{1/2}$$

are *equivalent*. These norms are associated to the inner products, respectively,

$$\langle f, g \rangle_1 = \int_\Omega f(x) g(x) + \nabla f(x) \cdot \nabla g(x)\, dx$$

and

$$(f, g)_1 = \int_\Omega \nabla f(x) \cdot \nabla g(x)\, dx.$$

Thus, for all practical purposes we are free to choose which one of these norms (or inner products) we want to use on $H_0^1(\Omega)$, assuming Ω is bounded. This will be used again and again in our applications.

2.4. **Weak solutions of Poisson's equation.** Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. We consider the Dirichlet problem for Poisson's equation,

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where

$$f \in L^2(\Omega)$$

is given.

Because of the Dirichlet boundary condition, the space $H_0^1(\Omega)$ is a natural place to look for solutions of this problem. Indeed, if $u \in H_0^1(\Omega)$, then the boundary condition is already satisfied, so we only have to worry about satisfying the equation $\Delta u = f$, in a weak sense. To find the correct weak formulation, let us first assume that u is classical solution. Then performing one integration by parts, we find that for every test function $v \in C_0^\infty(\Omega)$,

$$\int_\Omega \Delta u(x) v(x)\, dx = -\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

But the right hand side makes sense for all $u \in H^1(\Omega)$, since such a u has weak first order partial derivatives in $L^2(\Omega)$. Also, the left hand side should equal $\int_\Omega f v\, dx$. Therefore, we arrive at the following:

Definition 7. We say that $u \in H_0^1(\Omega)$ is a *weak solution* of the Dirichlet problem (2) if

$$-\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx \qquad \text{for all } v \in C_0^\infty(\Omega).$$

Note that this can be restated as:

$$(u, v)_1 = -F(v) \qquad \text{for all } v \in C_0^\infty(\Omega),$$

with notation as in §2.3, and with

$$F(v) = \int_\Omega f(x) v(x)\, dx = \langle f, v \rangle.$$

By the Cauchy-Schwarz inequality in $L^2(\Omega)$,

$$|F(v)| \le C \|v\|_2 \le C \|v\|_{1,2} \qquad (C = \|f\|_2),$$

so F is a bounded linear functional on $H_0^1(\Omega)$. But then, applying the Riesz representation theorem with $(\cdot,\cdot)_1$ as the inner product on $H_0^1(\Omega)$ (cf. §2.3), and using also the symmetry of the inner product, we immediately conclude:

Theorem 6. *For any $f \in L^2(\Omega)$, the Dirichlet problem* (2) *has a unique weak solution $u \in H_0^1(\Omega)$.*

We shall not here consider the more difficult problem of regularity, but let it be said that if $f \in C^\infty(\overline{\Omega})$, then also $u \in C^\infty(\overline{\Omega})$. See chapter 8 of McOwen for this.

Next, consider more general Dirichlet boundary conditions:

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Let us assume that g extends to a C^2 function on $\overline{\Omega}$, i.e., assume $g \in C^2(\overline{\Omega})$. Then changing variables to

$$w = u - g,$$

we transform (3) to a problem of the same form as (2), namely

$$\begin{cases} \Delta w = \tilde{f} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where

$$\tilde{f} = f - \Delta g.$$

By Theorem 6, the latter problem has a weak solution $w \in H_0^1(\Omega)$. Changing variables back, we then get $u = w + g$ as a weak solution of the original problem (3).

2.5. **More general elliptic equations.** The method from §2.4 can be applied also to the Dirichlet problem for more general elliptic equations. Specifically, consider a second order differential operator of *divergence form*, i.e.,

$$Lu(x) = \sum_{i,j=1}^{n} \partial_i \big(a_{ij}(x)\partial_j u(x)\big) + c(x)u(x),$$

where $a_{ij}, c \in C^\infty(\overline{\Omega})$. Moreover, we impose the *symmetry* assumption

$$a_{ij} = a_{ji} \qquad (i, j = 1, \dots, n), \tag{5}$$

and the *uniform ellipticity* assumption

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \varepsilon |\xi|^2 \qquad \text{for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^n, \tag{6}$$

for some constant $\varepsilon > 0$.

Remark 7. The Laplace operator Δ corresponds to the special case where (a_{ij}) is the identity matrix and $c = 0$.

Now consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $f \in L^2(\Omega)$ is given.

We would like to solve this by the same method that was used for Poisson's equation in §2.4. Starting from a smooth solution and integrating by parts, we see that the natural definition of a weak solution is the following:

Definition 8. We say that $u \in H_0^1(\Omega)$ is a *weak solution* of (7) if

$$\int_\Omega \left(-\sum_{i,j=1}^{n} a_{ij}(x)\partial_i u(x)\partial_j v(x) + c(x)u(x)v(x) \right) dx = \int_\Omega f(x)v(x)\,dx$$

for all $v \in C_0^\infty(\Omega)$.

It will be convenient to restate the condition for a weak solution as follows:

$$B(u,v) = -F(v) \qquad \text{for all } v \in C_0^\infty(\Omega), \tag{8}$$

where

$$B(u,v) = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j v(x) - c(x)u(x)v(x) \right) dx,$$

$$F(v) = \int_\Omega f(x)v(x)\,dx.$$

We already know from §2.4 that F is a bounded linear functional on $H_0^1(\Omega)$, so if it happens that $B(u,v)$ is an inner product on $H_0^1(\Omega)$ whose associated norm is equivalent to the standard norm on $H_0^1(\Omega)$, then existence follows immediately from the Riesz representation theorem.

So we need to investigate under which conditions $B(u,v)$ meets these criteria. First, it is clear that $B(u,v)$ is linear in both u and v. Second, by the symmetry assumption (5), we evidently have

$$B(u,v) = B(v,u).$$

The only remaining condition for $B(u,v)$ to be an inner product is that $B(u,u) \geq 0$, with equality if and only if $u = 0$. But by the ellipticity assumption (6),

$$\begin{aligned} B(u,u) &\geq \int_\Omega \left(\varepsilon \, |\nabla u(x)|^2 - c(x)\,|u(x)|^2 \right) dx \\ &\geq \int_\Omega \left(\varepsilon \, |\nabla u(x)|^2 - \gamma\,|u(x)|^2 \right) dx \\ &= \varepsilon \,\|\nabla u\|_{L^2(\Omega)}^2 - \gamma \,\|u\|_{L^2(\Omega)}^2, \end{aligned}$$

where

$$\gamma = \max_{x\in\overline{\Omega}} c(x).$$

Therefore, if $\gamma \leq 0$, we have $B(u,u) \geq \varepsilon \,\|\nabla u\|_{L^2(\Omega)}^2$. On the other hand, if $\gamma > 0$, then applying Poincaré's inequality (Theorem 5) gives $B(u,u) \geq (\varepsilon - C^2\gamma)\,\|\nabla u\|_{L^2(\Omega)}^2$, where C is the constant in Poincaré's inequality.

Therefore, if we impose the condition

$$\gamma = \max_{x\in\overline{\Omega}} c(x) < \frac{\varepsilon}{C^2}, \tag{9}$$

then for some $\delta > 0$,

$$B(u,u) \geq \delta \,\|\nabla u\|_{L^2(\Omega)}^2 = \delta\,|u|_{1,2}^2, \tag{10}$$

for all $u \in H_0^1(\Omega)$; here we use the notation introduced §2.3. (Specifically, $\delta = \varepsilon$ if $\gamma \leq 0$, and $\delta = \varepsilon C^2 - \gamma$ if $\gamma > 0$.)

Next, we estimate $B(u,u)$ from above. Since the coefficients a_{ij} and c are bounded (being continuous functions on the compact set $\overline{\Omega}$), there exists $M > 0$ such that

$$B(u,u) \leq M\,\|u\|_{1,2}^2, \tag{11}$$

for all $u \in H_0^1(\Omega)$.

Since $|\cdot|_{1,2}$ and $\|\cdot\|_{1,2}$ are equivalent norms on $H_0^1(\Omega)$ (cf. §2.3), we can conclude from (10) and (11) that these norms are also equivalent to the norm associated to the inner product $B(u,v)$, assuming (9) is satisfied.

Appealing to the Riesz representation theorem, we thus obtain:

Theorem 7. *If* (9) *is satisfied, then* (7) *has a unique weak solution* $u \in H_0^1(\Omega)$.

Remark 8. We are not saying that the condition (9) is necessary, only that it is sufficient. However, some kind of condition on $c(x)$ is certainly necessary, as there are choices of $c(x)$ such that (7) is not solvable for all f. For example, this is the case if $c(x)$ is constant and equals one of the Dirichlet eigenvalues of $-L$. In fact, condition (9) says that γ/ε is less than the smallest eigenvalue of $-L$ (which is positive). (Cf. Section 7.2.b in McOwen for the case $L = \Delta$.)

Remark 9. Even more general operators L can be treated if we use the Lax-Milgram lemma instead of the Riesz representation theorem. See chapter 6 of McOwen.