Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2007

Supplementary notes 2

Here we prove the Proposition in Section 4.2c of McOwen (with an alternative—and correct—proof):

Proposition (Regularity of the potential). *Assume* $f \in L^1(\Omega)$, *where* $\Omega \subset \mathbb{R}^n$ *is a bounded open set, and define*

$$u(x) = \int_\Omega K(x-y) f(y)\,dy \tag{1}$$

for $x \in \mathbb{R}^n$ *such that the integral exists (note that it certainly exists for* $x \in \mathbb{R}^n \setminus \overline{\Omega}$*). Here* K *is the usual fundamental solution of* Δ *on* $\mathbb{R}^n$*. The following regularity statements hold for* u*:*

(i) $u \in C^\infty\big(\mathbb{R}^n \setminus \overline{\Omega}\big)$*, and* u *is harmonic in* $\mathbb{R}^n \setminus \overline{\Omega}$.

(ii) If f *is bounded on* Ω*, then* u *is defined everywhere on* $\mathbb{R}^n$*, and* $u \in C^1(\mathbb{R}^n)$.

(iii) If $f \in C^1\big(\overline{\Omega}\big)$*, then* $u \in C^2(\Omega)$*. [Hence* $u \in C^2(\Omega) \cap C^1\big(\overline{\Omega}\big)$*, using also part (ii).]*

Remark 1. The space $L^1(\Omega)$ can be defined in two different ways, depending on whether one knows about Lebesgue integration or not. In the Lebesgue theory, it is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega |f(x)|\,dx < \infty$; such functions are called *integrable*. If instead we want to refer only to the Riemann integral, we could, as a replacement, define $L^1(\Omega)$ as the space of bounded functions $f : \Omega \to \mathbb{R}$ such that the set $\{x \in \Omega : f \text{ is discontinuous at } x\}$ has measure zero. Then if we assume additionally that Ω has smooth boundary (hence the boundary has measure zero), it is guaranteed that the Riemann integrals $\int_\Omega f(x)\,dx$ and $\int_\Omega |f(x)|\,dx$ exist.

Remark 2. Extend f by zero outside Ω; this extension, which we still denote f, is then a compactly supported distribution on $\mathbb{R}^n$, so by the general theory in Section 2.3.d, we know that $u = K * f$ is a solution of $\Delta u = f$ in $\mathscr{D}'(\mathbb{R}^n)$. It is therefore not surprising that u is harmonic outside $\overline{\Omega}$ (cf. part (i) of the Proposition).

In the proof of the proposition we shall use the following facts.

Fact 1. Suppose $g \in C^1(E)$, where $E \subset \mathbb{R}^n$ is an open set. Suppose further that a and b are two points in E such that the line segment between a and b is completely contained in E, i.e.,

$$\{a + t(b-a) : t \in [0,1]\} \subset E.$$

Then

$$g(b) - g(a) = \int_0^1 \nabla g\big(a + t(b-a)\big) \cdot (b-a)\,dt.$$

To prove this, just use the fundamental theorem of Calculus to write

$$g(b) - g(a) = \int_0^1 \frac{d}{dt}\left[g\big(a + t(b-a)\big)\right] dt,$$

and notice that by the chain rule, $\frac{d}{dt}\left[g\big(a + t(b-a)\big)\right] = \nabla g\big(a + t(b-a)\big) \cdot (b-a)$.

Fact 2. The following estimates hold:

$$\int_{|z|\le r} |K(z)|\,dz \le \begin{cases} Cr^2(1+|\log r|) & \text{if } n=2, \\ Cr^2 & \text{if } n\ge 3, \end{cases} \tag{2}$$

$$\int_{|z|\le r} |\nabla K(z)|\,dz \le Cr, \tag{3}$$

$$\int_{|z|=r} |K(z)|\,dS(z) \le \begin{cases} Cr(1+|\log r|) & \text{if } n=2, \\ Cr & \text{if } n\ge 3, \end{cases} \tag{4}$$

$$\int_{|z|=r} |\nabla K(z)|\,dS(z) \le C, \tag{5}$$

where $C = C(n)$ denotes constants which only depend on the dimension n. (Note that in the Riemann theory of integration, these integrals are improper.)

Let us prove these for $n=3$. Then $K(x) = -\frac{1}{4\pi|x|}$ and $\nabla K(x) = \frac{1}{4\pi|x|^2}\frac{x}{|x|}$, and integrating in spherical coordinates we get

$$\int_{|z|\le r} |K(z)|\,dz \le \int_0^r \int_{|y|=1} \frac{1}{4\pi\rho}\,dS(y)\,\rho^2\,d\rho = \int_0^r \rho\,d\rho = \frac{1}{2}r^2,$$

and

$$\int_{|z|\le r} |\nabla K(z)|\,dz \le \int_0^r \int_{|y|=1} \frac{1}{4\pi\rho^2}\,dS(y)\,\rho^2\,d\rho = \int_0^r d\rho = r.$$

Also,

$$\int_{|z|=r} |K(z)|\,dS(z) \le \frac{1}{4\pi r}\int_{|z|=r} dS(z) = \frac{1}{4\pi r}(4\pi r^2) = r,$$

and

$$\int_{|z|=r} |\nabla K(z)|\,dS(z) \le \frac{1}{4\pi r^2}\int_{|z|=r} dS(z) = \frac{1}{4\pi r^2}(4\pi r^2) = 1.$$

We leave the calculations in other dimensions as exercises.

Proof of part (i). We claim that for any multi-index α,

$$\partial^\alpha u(x) = \int_\Omega (\partial^\alpha K)(x-y) f(y)\,dy \qquad \text{for all } x \in \mathbb{R}^n \setminus \overline{\Omega}. \tag{6}$$

Let us show this for $\alpha = e_j$, the j-th standard basis vector; the general case follows by the same argument, used repeatedly. First, however, let us note that (6) implies $\Delta u(x) = \int_\Omega (\Delta K)(x-y) f(y)\,dy = 0$ for $x \in \mathbb{R}^n \setminus \overline{\Omega}$, since $\Delta K = 0$ away from the origin.

So now fix $x \in \mathbb{R}^n \setminus \overline{\Omega}$. For $h \ne 0$ define

$$A_h = \frac{u(x+he_j) - u(x)}{h} - \int_\Omega (\partial_j K)(x-y) f(y)\,dy.$$

Fix $\varepsilon > 0$. We have to prove there exists $\delta > 0$ such that $|A_h| \le \varepsilon$ for all $0 < |h| \le \delta$. Then it follows that

$$\partial_j u(x) = \int_\Omega (\partial_j K)(x-y) f(y)\,dy, \tag{7}$$

as desired.

Since $\mathbb{R}^n \setminus \overline{\Omega}$ is an open and nonempty set, we can find $r > 0$ such that $B_r(x) \subset \mathbb{R}^n \setminus \overline{\Omega}$. Thus, $y \in \Omega \implies |x-y| \ge r$. By the triangle inequality, this gives[1]

$$y \in \Omega,\ |h| \le r/2 \implies |x + he_j - y| \ge r/2. \tag{8}$$

[1] We have $|x+he_j-y| \ge |x-y| - |he_j| = |x-y| - |h| \ge r - r/2 = r/2$.

Also, we choose $R > 0$ so large that $|x| + |y| + r/2 \le R$ for all $y \in \Omega$; we can do this because Ω is bounded. So then we have $|x + he_j - y| \le R$ for all $y \in \Omega$ and all $|h| \le r/2$, again using the triangle inequality.

Since $\partial_j K(z)$ is uniformly continuous in the set $r/2 \le |z| \le R$, we can now conclude that, given any $\varepsilon' > 0$, there exists $\delta > 0$ (satisfying also $\delta \le r/2$) such that

(9) $$\left|(\partial_j K)(x + the_j - y) f(y) - (\partial_j K)(x - y)\right| \le \varepsilon' \qquad \text{for all } |h| \le \delta \text{ and all } y \in \Omega.$$

In fact, we choose $\varepsilon' > 0$ so small that

(10) $$\varepsilon' \int_\Omega |f(y)|\, dy \le \varepsilon.$$

Using Fact 1 we can write

$$\frac{u(x + he_j) - u(x)}{h} = \int_\Omega \frac{K(x + he_j - y) - K(x - y)}{h} f(y)\, dy = \int_\Omega \int_0^1 (\partial_j K)(x + the_j - y) f(y)\, dt\, dy,$$

hence for all $0 < |h| \le \delta$, using also (9) and (10), we get

$$\begin{aligned} |A_h| &= \left| \int_\Omega \int_0^1 \left[(\partial_j K)(x + the_j - y) f(y) - (\partial_j K)(x - y) \right] f(y)\, dt\, dy \right| \\ &\le \int_\Omega \int_0^1 \left|(\partial_j K)(x + the_j - y) f(y) - (\partial_j K)(x - y)\right| |f(y)|\, dt\, dy \\ &\le \varepsilon' \int_\Omega \int_0^1 |f(y)|\, dt\, dy = \varepsilon' \int_\Omega |f(y)|\, dy \le \varepsilon, \end{aligned}$$

and we are done. □

Proof of part (ii). Now we have the additional assumption that $|f(x)| \le M$ for all $x \in \Omega$. Then $u(x)$ is defined everywhere (note that in the Riemann theory of integration, (1) will be an improper integral if $x \in \overline{\Omega}$, but a convergent one).

To prove that $u \in C^1(\mathbb{R}^n)$ we would like to proceed more or less as in the proof of part (i), to show that (7) holds for all $x \in \mathbb{R}^n$, but we need to modify the argument slightly to handle the singularity in $K(x - y)$ when y approaches x (this can happen if $x \in \overline{\Omega}$), since then the uniform continuity fails. But this is not hard: we apply the usual trick of cutting out a small ball around the singularity, and consider that part of the integral separately, using the estimates in Fact 2.

So fix $x \in \overline{\Omega}$ (the set $\mathbb{R}^n \setminus \overline{\Omega}$ is covered by part (i) already), and fix $\varepsilon > 0$. Define A_h as before, and write

$$A_h = \int_\Omega \int_0^1 \left[(\partial_j K)(x + the_j - y) f(y) - (\partial_j K)(x - y) \right] f(y)\, dt\, dy = I_h + J_h,$$

where

$$\begin{aligned} I_h &= \int_{\Omega \cap \overline{B_r(x)}} \int_0^1 \left[(\partial_j K)(x + the_j - y) f(y) - (\partial_j K)(x - y) \right] f(y)\, dt\, dy, \\ J_h &= \int_{\Omega \setminus \overline{B_r(x)}} \int_0^1 \left[(\partial_j K)(x + the_j - y) f(y) - (\partial_j K)(x - y) \right] f(y)\, dt\, dy, \end{aligned}$$

where the small number $r > 0$ will be chosen in a moment; r will depend on ε.

We now estimate, assuming $0 < |h| \le r$,

$$\begin{aligned} |I_h| &\le M \int_{B_r(x)} \int_0^1 \left[|(\partial_j K)(\underbrace{x + the_j - y}_{=z})| + |(\partial_j K)(\underbrace{x - y}_{=z'})| \right] dt\, dy \\ &\le M \left[\int_{|z| \le 2r} \int_0^1 |(\partial_j K)(z)|\, dt\, dz + \int_{|z| \le r} \int_0^1 |(\partial_j K)(z')|\, dt\, dz' \right] \\ &\le 2M \int_{|z| \le 2r} |(\partial_j K)(z)|\, dz \le 4MCr, \end{aligned}$$

where in the last step we used (3) from Fact 2. We now choose r such that $4MCr = \varepsilon/2$, i.e.,

$$r = \frac{\varepsilon}{8MC}.$$

So with this choice, we have $|I_h| \le \varepsilon/2$ for all $0 < |h| \le r$.

Having fixed r, we now observe that part (i), with Ω replaced by $\Omega \setminus \overline{B_r(x)}$, gives us: There exists $\delta > 0$ (which we can assume is $\le r$) such that $|J_h| \le \varepsilon/2$ for all $0 < |h| \le \delta$.

We conclude that

$$|A_h| \le |I_h| + |J_h| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{whenever } 0 < |h| \le \delta,$$

and this proves (7). We leave it as an exercise to prove that $\partial_j u$ as given by (7) is in fact a continuous function on $\mathbb{R}^n$. □

Proof of part (iii). Now we assume $f \in C^1(\overline{\Omega})$. In particular, f and ∇f are both bounded functions, so we can find $M > 0$ such that $|f(x)| \le M$ and $|\nabla f(x)| \le M$ for all $x \in \overline{\Omega}$.

Note that the proof in McOwen is flawed, since the integral on the left side at the bottom of p. 115 is divergent for all $x \in \Omega$.

Instead, we argue as follows, to prove that $u \in C^2(\Omega)$. First, by part (ii) we already know that $u \in C^1(\mathbb{R}^n)$, and that for all x,

$$\partial_j u(x) = \int_\Omega (\partial_j K)(x-y) f(y)\, dy. \tag{11}$$

Now fix $x \in \Omega$. We would like to integrate by parts in the above integral, to get the derivative onto f. As usual, to avoid the singularity, we cut out a small ball around x. So we write

$$\int_\Omega (\partial_j K)(x-y) f(y)\, dy = \int_{B_\varepsilon(x)} (\partial_j K)(x-y) f(y)\, dy + \int_{\Omega \setminus B_\varepsilon(x)} (-1) \frac{\partial}{\partial y_j} \left[K(x-y) \right] f(y)\, dy \equiv I_\varepsilon + J_\varepsilon,$$

for any $\varepsilon > 0$ so small that $B_\varepsilon(x) \subset \Omega$. Using (3) from Fact 2, we see that

$$|I_\varepsilon| \le MC\varepsilon.$$

Integrating by parts in J_ε we get

$$J_\varepsilon = \int_{\Omega \setminus B_\varepsilon(x)} K(x-y) \partial_j f(y)\, dy + \int_{|y-x|=\varepsilon} K(x-y) f(y) \nu_j \, dS(y) \equiv J_\varepsilon^{(1)} + J_\varepsilon^{(2)},$$

where $\nu = (y-x)/|y-x|$ is the outward unit normal on the sphere $B_\varepsilon(x)$. By (4) from Fact 2 (here the log can be removed if $n \ge 3$),

$$\left| J_\varepsilon^{(2)} \right| \le MC\varepsilon \left(1 + \left| \log \varepsilon \right| \right).$$

Finally,

$$J_\varepsilon^{(1)} = \int_\Omega K(x-y) \partial_j f(y)\, dy - \int_{B_\varepsilon(x)} K(x-y) \partial_j f(y)\, dy,$$

and by (2) from Fact 2,

$$\left| \int_{B_\varepsilon(x)} K(x-y) \partial_j f(y)\, dy \right| \le MC\varepsilon^2 \left(1 + \left| \log \varepsilon \right| \right).$$

Combining the above estimates, and letting $\varepsilon \to 0$, we conclude that

$$\partial_j u(x) = \int_\Omega (\partial_j K)(x-y) f(y)\, dy = \int_\Omega K(x-y) \partial_j f(y)\, dy.$$

But now we can apply part (ii) to the integral on the right, and conclude that $u \in C^2(\Omega)$, with

$$\partial_k \partial_j u(x) = \int_\Omega (\partial_k K)(x-y) \partial_j f(y)\, dy.$$

(We leave it as an exercise to show that this is a continuous function of x in Ω.) □