Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2007

Supplementary Problem Set 1

1 Solve using the method of characteristics:

(a) $xu_x + yu_y = 2u$, $u(x,1) = g(x)$.

(b) $uu_x + u_y = 1$, $u(x,x) = x/2$.

2 Solve Burgers' equation

$$u_t + uu_x = 0$$

for $t > 0$, with initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 0, \\ 2 & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Moreover, we require that u satisfy the condition $u_l > u_r$ across a shock. (This is called the entropy condition, and it can be justified on physical grounds. It also ensures uniqueness of the solution. Cf. Section 1.2.b in McOwen, and the Remark at the end of that section.)

Draw a picture of the shocks and characteristics in the (x,t)-plane.

3 Let $u(x,t)$ be the solution to the Cauchy problem

$$u_t + cu_x + u^2 = 0, \qquad u(x,0) = x,$$

where c is a constant.

(a) Solve the problem.

(b) A person leaves the point x_0 at time $t = 0$, and moves in the positive x-direction with speed c (i.e., the quantity $x - ct$ is fixed for him). Show that if $x_0 > 0$, then the solution as seen by this person approaches zero as $t \to \infty$.

(c) What will be observed by such a person if $x_0 < 0$ or $x_0 = 0$?

4 (a) Show that the following equation is hyperbolic:

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0.$$

(b) Transform the equation to canonical coordinates.

(c) Find the general solution $u(x,y)$.

(d) Find a solution that satisfies $u(-x,2x) = x$ and $u(x,0) = \sin 2x$.

5 Solve the problem

$$u_{tt} - 4u_{xx} = e^x + \sin t, \qquad u(x,0) = 0, \qquad u_t(x,0) = \frac{1}{1+x^2},$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}$.

6 Do Exercise 2.3.16 from McOwen.

7 The purpose of this exercise is to prove that every linear ordinary differential operator with constant coefficients,

$$L = \sum_{j=0}^{k} c_j \left(\frac{d}{dx}\right)^j,$$

has a fundamental solution. Here the c_j are constants, and we assume $c_k \neq 0$ (so L is genuinely k-th order).

Let v be the solution of $Lv = 0$ with $v(0) = \cdots = v^{(k-2)}(0) = 0$ and $v^{(k-1)}(0) = c_k^{-1}$. (This solution exists, by ODE theory.)

Now define $F(x) = v(x)$ for $x > 0$ and $F(x) = 0$ for $x < 0$. Prove that $LF = \delta$, i.e., F is a fundamental solution.

8 Show that the characteristic function of the first quadrant in the (x, y)-plane (i.e., $F(x, y) = 1$ if $x, y > 0$ and $F(x, y) = 0$ otherwise) is a fundamental solution for $\partial_x\partial_y$ in $\mathbb{R}^2$.

9 Do Exercise 4.1.8 from McOwen.

10 Suppose Ω is a bounded domain with smooth boundary, and suppose

$$u \in C^2\big(\Omega \times (0,T)\big) \cap C^1\big(\overline{\Omega} \times (0,T)\big)$$

satisfies

$$u_t = \Delta u \qquad (x \in \Omega, 0 < t < T),$$

with either $u = 0$ or $\partial u/\partial \nu = 0$ on the boundary $\partial\Omega$, for all $0 < t < T$. Define

$$f(t) = \int_\Omega u(x,t)^2\,dx \qquad (0 < t < T).$$

Prove that $f(t)$ is nonincreasing.

Hint: Show that $u(u_t - \Delta u) = \frac{1}{2}\partial_t\big(u^2\big) - \operatorname{div}(u\nabla u) + |\nabla u|^2$, and integrate over Ω.

11 (a) Show that the general radial solution to the 3d wave equation (with $c = 1$) is

$$u(x,t) = \frac{1}{r}\left[\phi(r+t) + \psi(r-t)\right] \qquad (r = |x|),$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are arbitrary.

(b) Solve the Cauchy problem for the 3d wave equation with radial data:

$$u_{tt} - \Delta u = 0, \qquad u(x,0) = f(|x|), \qquad u_t(x,0) = g(|x|),$$

where f, g are defined on $[0,\infty)$. (*Hint:* Find a formula similar to the d'Alembert formula; first extend f and g to even functions on $\mathbb{R}$.)

(c) Let u, f, g be as in part (b). Show that $u(0,t) = f(t) + tf'(t) + tg(t)$. Thus, u is generally no better than C^k if $f \in C^{k+1}$ and $g \in C^k$.

$\boxed{12}$ Show that $K(x) = -\dfrac{e^{-c|x|}}{4\pi|x|}$ is a fundamental solution for $\Delta - c^2$ on $\mathbb{R}^3$.

$\boxed{13}$ In this exercise, A, B, C and R denote real $n \times n$-matrices.

We say that $A = (a_{ij})$ is *positive definite* (resp. *negative definite*) if $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j > 0$ (resp. < 0) for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$. If this property holds with the sharp inequalities replaced by $\geq$ (resp. $\leq$), then A is said to be *positive semi-definite* (resp. *negative semi-definite*).

(a) Show that if A is positive semi-definite, then $a_{ii} \geq 0$ for $i = 1, \dots, n$. Moreover, if λ is an eigenvalue of A, then $\lambda \geq 0$.

(b) Prove that if A and B are symmetric and positive semi-definite, then $\operatorname{tr}(AB) \geq 0$, where tr denotes the trace. (*Hint:* Diagonalize A using an orthonormal basis of eigenvectors. Use part (a) and the fact that $\operatorname{tr}(R^tCR) = \operatorname{tr}(C)$ for all C if R is an orthogonal matrix.)

$\boxed{14}$ Let $u : \Omega \to \mathbb{R}$ be C^2. Prove that if u has a local maximum at at point $x_0 \in \Omega$, then the symmetric $n \times n$-matrix $D^2u(x_0)$ with entries $\partial_i\partial_j u(x_0)$ is negative semi-definite. (*Hint:* Given $\xi \in \mathbb{R}^n$, $\xi \neq 0$, define $\phi(t) = u(x_0 + t\xi)$ for t in a small interval around 0.)

$\boxed{15}$ The purpose of this exercise is to prove the weak maximum principle (cf. (16) in Section 4.1 of McOwen) for a more general elliptic operator than the Laplace operator. So let Ω be a bounded domain in $\mathbb{R}^n$, and let

$$L = \sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^n b_i(x)\partial_i,$$

where a_{jk} and b_j are continuous functions on $\overline{\Omega}$ and the matrix (a_{ij}) is symmetric (so $a_{jk} = a_{kj}$) and positive definite, i.e.,

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > 0 \qquad \text{for all } x \in \overline{\Omega} \text{ and all } \xi \in \mathbb{R}^n \text{ with } \xi \neq 0. \tag{1}$$

(Thus, the operator L is elliptic.)

(a) Show that if $v \in C^2(\Omega)$ satisfies $Lv > 0$ in Ω, then v cannot have a local maximum in Ω. (*Hint:* Use the two previous problems to get a contradiction if we assume that a local maximum exists.)

(b) Show that if $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and $M > 0$ is sufficiently large, then $w(x) = \exp(-M|x - x_0|^2)$ satisfies $Lw > 0$ in Ω.

(c) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and that $Lu = 0$ in Ω. Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(*Hint:* Show that this conclusion holds for $v = u + \varepsilon w$, where w is as above and $\varepsilon > 0$.)