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TMA4305 PARTIAL DIFFERENTIAL EQUATIONS

Engelsk
Lørdag 27. mai 2006
kl. 9–13

Hjelpemidler (kode C): Typegodkjent kalkulator med tomt minne (HP 30S), samt ett A4-ark stemplet av Institutt for matematiske fag, med valgfri påskrift av studenten.

Sensurdato: 16. juni 2005

All answers must be justified.

Problem 1

a) Formulate (without proof) the weak maximum principle for the heat equation

$$u_t = u_{xx} \qquad \text{in the region } a < x < b,\ 0 < t < T.$$

b) We now consider the equation (with a variable coefficient)

(1) $$u_t = x u_{xx} \qquad \text{in the region } -2 < x < 2,\ 0 < t < 1.$$

Verify that $u = -2xt - x^2$ is a solution. Is the weak maximum principle valid for (1)?

Problem 2 Find the entropy solution of Burgers' equation

$$u_t + u u_x = 0 \qquad \text{with initial condition} \qquad u(x,0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } 0 \le x \le 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Also, sketch the characteristics and shock curves (if any) in the xt-plane.

Problem 3 Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.

Prove uniqueness of smooth solutions of the initial/boundary value problem (with Neumann boundary condition)

$$\begin{cases} u_{tt} = \Delta u & \text{in } U \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \partial U \times [0,T], \\ u = f, \qquad u_t = g & \text{on } U \times \{t = 0\}, \end{cases}$$

where $f, g \in C^\infty(\overline{U})$ and $h \in C^\infty(\partial U \times [0,T])$ are given functions, and ν is the outward pointing unit normal vector on the boundary of U. (*Hint:* Energy method.)

Problem 4 Find the solution of the initial value problem

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz} \\ u(x,y,z,0) = x^2 + y^2, \qquad u_t(x,y,z,0) = 0. \end{cases}$$

Problem 5

Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.

a) Formulate the definition of a weak solution $u \in H_0^1(U)$ of the problem

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2}$$

where $f \in L^2(U)$ is a given function. Prove the existence of a weak solution.

b) Suppose $u \in H_0^1(U)$ is a weak solution of (2) (with $f \in L^2(U)$), and suppose further that u has compact support in U (i.e., supp $u \subset\subset U$). Show that

$$u \in H^2(U).$$