Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2008

Problem Set for Week 17

1 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\|\cdot\|_{2,2}$ denote the norm in $H^2(\Omega)$, and

$$H_0^2(\Omega) = \left\{ u \in H^2(\Omega) : \text{there is a sequence } \{\phi_i\} \subset C_0^\infty(\Omega) \text{ such that } \lim \|\phi_i - u\|_{2,2} = 0 \right\}.$$

$H_0^2(\Omega)$ is a closed subspace of $H^2(\Omega)$ and hence Hilbert space with the H^2-inner product. Prove the following Poincare type inequality:

$$\|u\|_2 \le C_\Omega \|\Delta u\|_2 \qquad \forall u \in H_0^2(\Omega),$$

where C_Ω is a constant depending only on Ω.

Hint: See the proof of Theorem 1 page 173 in McOwen. Prove the inequalities for $u \in C_0^\infty(\Omega)$, use density.

2 (Evans 6.6:2) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A function $u \in H_0^2(\Omega)$ is a weak solution of the following boundary value problem for the *biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{in} \quad \partial\Omega, \end{cases} \tag{1}$$

if

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^2(\Omega).$$

If $f \in L^2(\Omega)$, prove that exists a unique weak solution of (1).

Hint: Riesz representation theorem. Exercise 1 may be usefull.

3 Exercise 6.2:4 in McOwen.

4 Exercise 6.3:3 in McOwen.

5 Exercise 6.3:7 in McOwen.