Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations
Spring 2008

Problem Set for Week 18

1 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $g \in H^1(\Omega)$, and define

$$\mathscr{A} = \left\{u \in H^1(\Omega) : u - g \in H_0^1(\Omega)\right\}.$$

a) Show that $\mathscr{A}$ is a weakly closed subset of $H^1(\Omega)$, i.e. show that

$$\mathscr{A} \ni u_j \rightharpoonup u \quad \text{in} \quad H^1(\Omega) \qquad \Rightarrow \qquad u \in \mathscr{A}.$$

Hint: $H_0^1(\Omega) \ni u_j - g \rightharpoonup u - g$ in $H^1(\Omega)$.

Define $F : H^1(\Omega) \to \mathbb{R}$ by

$$F(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + f u\right) dx.$$

b) Prove that F is coercive on $\mathscr{A}$, i.e. there are constants $C_1 > 0, C_2 \geq 0$ such that

$$F(u) \geq C_1 \|u\|_{1,2}^2 - C_2 \quad \text{for all} \quad u \in \mathscr{A}.$$

Hint: Prove that in any normed space $(X, \|\cdot\|)$,

$$\|x - y\|^2 \geq \frac{1}{2}\|x\|^2 - \|y\|^2 \qquad \text{and} \qquad \|x\|^2 \geq \frac{1}{2}\|x - y\|^2 - \|y\|^2 \qquad \text{for all} \qquad x, y \in X.$$

Hint 2: Triangle inequality + $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$ for $a, b \geq 0$.

2 Exercise 7.1:6 in McOwen.

3 Exercise 7.1:8 b in McOwen.