TMA4305 Partial Differential Equations Spring 2008

Solutions week 15

1 (McOwen 5.1:2)

(1) $$\begin{cases} u_t &= \Delta u & \text{in } \Omega\times(0,\infty) \\ u &= h & \text{on } \partial\Omega\times(0,\infty) \\ u &= g & \text{on } \overline{\Omega}\times\{0\} \end{cases}$$

Note that $u(x,t) = v(x,t) + w(x)$ solve (1) if v and w solve

$$\begin{cases} \Delta w &= 0 & \text{in } \Omega \\ w &= h & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} v_t &= \Delta v & \text{in } \Omega\times(0,\infty) \\ v &= 0 & \text{on } \partial\Omega\times(0,\infty) \\ v &= g-w & \text{on } \overline{\Omega}\times\{0\} \end{cases}$$

If $\{\lambda_n,\varphi_n\}_{n=1}^{\infty}$ are eigenvalues and eigenfunctions for $-\Delta$ on Ω and $g-w=\sum a_n\varphi_n$, then from McOwen,

$$v(x,t) = \sum a_n e^{-\lambda_n t}\varphi_n.$$

Note that, for $t\to\infty$,

$$|v(x,t)| \le e^{-\lambda_i t}\Big|\sum_{n=1}^{\infty} a_n e^{-(\lambda_n-\lambda_i)t}\varphi_n\Big| \le e^{-\lambda_i t}\Big|\sum_{n=1}^{\infty} a_n\varphi_n\Big| \to 0$$

where λ_i is the smallest positive eigenvalue and we assumed $|\sum_{n=1}^{\infty} a_n\varphi_n|$ to be bounded. Hence

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}(v(x,t)+w(x)) = w(x)$$

2 (McOwen 5.2:1)

Theorem 1. *If g bounded continuous function and*

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t)g(y)\,dy$$

where

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$$

Then,

i) $u\in C^{\infty}(\mathbb{R}^n\times(0,\infty))$

ii) $u_t = \Delta u$ *in* $(\mathbb{R}^n\times(0,\infty))$

iii) $\lim_{t\to 0} u(x,t) = g(x)$

Proof.

(a) Obs: K is C^{∞} for $t>0$

$$K_t = -\frac{n}{2}\frac{4\pi}{(4\pi t)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4t}} + \frac{1}{(4\pi t)^{\frac{n}{2}}}\frac{|x-y|^2}{4t^2} e^{-\frac{|x-y|^2}{4t}}$$

$$K_{x_ix_i} = \frac{1}{(4\pi t)^{\frac{n}{2}}}[-\frac{2(x_i-y_i)}{4t}e^{-\frac{|x-y|^2}{4t}}]_{x_i}$$
$$= \frac{1}{(4\pi t)^{\frac{n}{2}}}(-\frac{2}{4t}+\frac{4(x_i-y_i)^2}{4t})e^{-\frac{|x-y|^2}{4t}}$$

So, $K_t - \sum_{i=1}^n K_{x_ix_i} = 0$ $(t>0)$

(b)

$$\int_{\mathbb{R}^n} K(x,y,t)dy = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dy \underset{z=\frac{y-x}{2\sqrt{t}}}{=} \frac{1}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|z|^2}dz = \frac{1}{\pi^{\frac{n}{2}}}\left(\int_{\mathbb{R}} e^{-s^2}ds\right)^n = 1$$

(c)

$$\int_{|x-y|>\delta} K(x,y,t)dy \underset{z=\frac{y-x}{2\sqrt{t}}}{=} \frac{1}{\pi^{\frac{n}{2}}}\int_{|z|>\frac{\delta}{2\sqrt{t}}} e^{-|z|^2}dz \le \frac{1}{\pi^{\frac{n}{2}}}\int_{|z|>\frac{\delta}{2\sqrt{t}}} e^{-\frac{1}{2}\frac{\delta^2}{4t}}e^{-\frac{1}{2}|z|} \le \frac{e^{-\frac{\delta^2}{8t}}}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}}dz \to 0$$

as $t \to 0$ uniformly in x since $\int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}}dz < \infty$.

(d)

(i)

$$D^\alpha u(x,t) = \int_{\mathbb{R}^n}(D^\alpha_{t,x}K)(x,y,t)g(y)dy$$

is continuous for all α, hence $u \in C^\infty(\mathbb{R}\times(0,\infty))$

(ii)

$$u_t - \Delta u = \int_{\mathbb{R}^n}(K_t(x,y,t)-\Delta_x K(x,y,t))g(y)dy = 0$$

for all $t>0$ since $K_t(x,y,t)-\Delta_x K(x,y,t) = 0$

(iii) Using (b),

$$u(x,t)-g(x) = \int_{\mathbb{R}^n} K(x,y,t)(g(y)-g(x))dy$$
$$= (\int_{|x-y|<\delta} + \int_{|x-y|>\delta})K(x,y,t)(g(y)-g(x))dy$$

that implies

$$|u(x,t)-g(x)| \le \int_{|x-y|<\delta} K(x,y,t)|g(y)-g(x)|dy + 2\|g\|_\infty \int_{|x-y|>\delta} K(x,y,t)dy$$

For all $\epsilon>0$ take $\delta>0$ such that $|x-y|<\delta$ implies $|g(x)-g(y)|<\frac{\epsilon}{2}$ (g continuous) and $t>0$ small such that $\int_{|x-y|>\delta} K(x,y,t)dy < \frac{\epsilon}{2}\frac{1}{2\|g\|_\infty}$ by (c). Then,

$$|u(x,t)-g(x)| < \epsilon$$

Remark: $(iii)+(i) \Rightarrow u \in C(\mathbb{R}^n\times[0,\infty))$.

□

3 (McOwen 5.2:2)

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t)g(y)dy$$

where g bounded and continuous.

(a)

$$|u(x,t)| \le \int_{\mathbb{R}^n} K(x,y,t)|g(y)|dy \quad \text{since } K>0$$
$$\le \|g\|_\infty \int_{\mathbb{R}^n} K(x,y,t)dy$$
$$= \|g\|_\infty \quad \text{since} \int K(x,y,t)dy = 1$$

(b) Assume in addition $\int_{\mathbb{R}^n} |g(y)|dy < \infty$, then

$$\begin{aligned} |u(x,t)| &\le \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} |g(y)|dy \\ &\le \frac{\int_{\mathbb{R}^n} |g(y)|dy}{(4\pi t)^{\frac{n}{2}}} \quad \text{since } e^{-\frac{|x-y|^2}{4t}} \le 1 \end{aligned}$$

and the last term goes to zero as $t \to \infty$.

4 (McOwen 5.2:5)

Theorem 2. *Assume $u \in C(U_T \cup \Gamma_T) \cap C^{2,1}(U_T) \cap L^\infty(U_T)$ and $u_t - \Delta u \le 0$ in $U_T := \Omega \times (0,T)$ where $\Gamma_T = \Omega \times \{0\} \bigcup \partial\Omega \times (0,T)$. Then,*

$$\sup_{U_T} u = \sup_{\mathbb{R}^n} u(x,0)$$

Proof.

1) Let $\tau < T$, $\epsilon > 0$, $k > 0$:

$$w(x,t) = u(x,t) - \epsilon|x|^2 - kt$$

Obs:

$$w_t - \Delta w \le 2n\epsilon - k < 0$$

if $k > 2n\epsilon$.

2) Obs:

$$\lim_{|x|\to\infty} w(x,t) = -\infty$$

Take $R > 0$ such that

$$|x| > R \Rightarrow \epsilon R^2 > 2\|u\|_\infty + kT + 1 \Rightarrow w(x,t) < -\|u\|_\infty - 1$$

On the other hand at $(x,t) = (0,0)$

$$w(0,0) = u(0,0) \ge -\|u\|_\infty$$

Conclusion:

$$\sup_{U_\tau} w = \sup_{B_R(0)\times[0,\tau)} u = \max_{\overline{B_R(0)\times[0,\tau]}} u$$

3) Let $(x,t) \in \overline{U}_\tau$ such that

$$w(x,t) = \max_{\overline{U}_T} w$$

If $0 < t < \tau$, then $w_t = 0$ and $\Delta w \le 0$. If $t = \tau$, then $w_t \ge 0$ and $\Delta w \le 0$. Both cases are in contradiction with the observation in (1). Hence,

$$\max_{\overline{U}_\tau} w = \max_{\mathbb{R}^n} w(x,0)$$

(4) Let $(x,t) \in U_T$: Then $(x,t) \in U_\tau$ for some $\tau < T$ and

$$u(x,t) = w(x,t) + \epsilon|x|^2 + kt \le \max_{\mathbb{R}^n} w(x,0) + \epsilon|x|^2 + kT \le \sup_{\mathbb{R}^n} u(x,0) + \epsilon|x|^2 + kT,$$

where the last inequality follows since $w \le u$. Send $\epsilon \to 0$, then $k \to 0$:

$$u(x,t) \le \sup_{\mathbb{R}^n} u(x,0)$$

and hence

$$\sup_{U_T} u \le \sup_{\mathbb{R}^n} u(x,0).$$

□

5 (McOwen 5.2:7 a)

$$\begin{cases} u_t &= u_{xx} & \text{in } (0,\infty)^2 \\ u &= g & \text{on } (0,\infty)\times\{0\} \\ u &= 0 & \text{on } \{0\}\times[0,\infty) \end{cases}$$

Assume g bounded continuous function such that $g(0)=0$. Idea: odd extension

$$\tilde{g}(x) = \begin{cases} g(x) & x>0 \\ -g(-x) & x<0 \end{cases}$$

Obs:

$$u = v|_{x>0}$$

where

$$\begin{cases} v_t &= v_{xx} & \text{in } \mathbb{R}\times(0,\infty) \\ v &= g & \text{on } \mathbb{R}\times\{0\} \end{cases}$$

Moreover, v odd and continuous, so

$$v(0,t) = 0 \quad \text{for all } t \geq 0$$

Formula

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}} K(x,y,t)\tilde{g}(y)dy \\ &= \frac{1}{(4\pi t)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\tilde{g}(y)dy \\ &= \frac{1}{(4\pi t)^{\frac{1}{2}}}\left[\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} - g(-y)dy + \int_0^{+\infty} e^{-\frac{(x-y)^2}{4t}} g(y)dy\right] \\ &= \frac{1}{(4\pi t)^{\frac{1}{2}}}\int_0^{+\infty}[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}]g(y)dy \end{aligned}$$

6 (McOwen 5.2:9 a)

Theorem 3. *Assume f f_t f_{x_i} $f_{x_ix_i}$ continuous bounded in $\mathbb{R}^n\times[0,\infty)$ and let*

$$u(x,t) = \int_0^t\int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)dyds$$

where

$$K(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

Then:

i) $u \in C^{2,1}(\mathbb{R}^n\times(0,\infty)) \cap C(\mathbb{R}^n\times[0,\infty))$
ii) $u_t - \Delta u = f$ *in* $\mathbb{R}^n\times(0,\infty)$
iii) $\lim_{t\to 0} u(x,t) = 0$ *in* $\mathbb{R}^n$

The original solution was incorrect!

The problem is that the integrand is singular. To interchange the order derivation and integration you must use the Lebesgue Dominated Convergence Theorem from Measure Theory. This has not been a part of the curruculum this year, therefore **this problem is not revant for the exam!**

A correct solution to this problem can be found on pages 50-51 in the book

L. C. Evans: *Partial differential equations*, AMS, 1997.