Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2008

Problem set for week 16

1 See solutions from last year (2007):
http://www.math.ntnu.no/emner/TMA4305/2007v/Week11.pdf

2 **a)** Let $\overline{\Omega}$ be a bounded set, $\left(C(\overline{\Omega})\,; \quad \|f\|_\infty = \max_{x\in\overline{\Omega}}|f(x)|\right)$ is a Banach space.

Proof.

1) $\|\cdot\|_\infty$ is well-defined, indeed:

$$f \in C(\overline{\Omega}) \Rightarrow \exists\, x_0 \in \overline{\Omega} \text{ such that } \|f\|_\infty = |f(x_0)|$$

2) $\|\cdot\|_\infty$ is a norm on $C(\overline{\Omega})$. Trivial, for example,

$$\|f\|_\infty = 0 \Rightarrow |f(x)| = 0 \text{ for all } x \Rightarrow f = 0$$

3) Completeness:

$$\{f_i\} \subset C(\overline{\Omega}) \text{ Cauchy} \Rightarrow f_i(x) \subset \mathbb{R} \text{ Cauchy},$$

since

$$|f_i(x) - f_j(x)| \le \|f_i - f_j\|_\infty.$$

This implies that there exists $y_x \in \mathbb{R}$ ($\mathbb{R}$ is complete) such that

$$f_i(x) \to y_x \quad \text{for all} \quad x \in \overline{\Omega}$$

Define the function f as

$$f(x) = y_x \quad \text{for all} \quad x \in \overline{\Omega},$$

and note that

$$\{f_i\} \text{ Cauchy} \quad \Rightarrow \quad \|f_i\|_\infty \le M < \infty\ \forall\, i \quad \Rightarrow \quad |f(x)| \le M\ \forall\, x \quad \Rightarrow \quad \sup_{\overline{\Omega}}|f(x)| \le M,$$

and

$$\sup_{\overline{\Omega}}|f_j(x) - f(x)| = \sup_{\overline{\Omega}}|f_j(x) - \lim_k f_k(x)| = \lim_k \sup_{\overline{\Omega}}|f_j(x) - f_k(x)| = \lim_k \|f_j - f_k\| \to 0$$

as $j \to 0$. Hence,

(i) $f_j \to f$ uniformly in $\overline{\Omega}$ implies $f \in C(\overline{\Omega})$

(ii) $\|f_j - f\|_\infty \to 0$ as $j \to 0$

and $C(\overline{\Omega})$ is complete.

□

b) Let Ω be a bounded set, then

$$\left(C^1(\overline{\Omega})\,; \quad \|f\|_{1,\infty} := \sup_{x\in\Omega}\left\{|f(x)| + |\nabla f(x)|\right\}\right)$$

is a Banach space.

Proof.

1) $\|\cdot\|_{1,\infty}$ well defined and norm on $C^1(\overline{\Omega})$ as in the previous exercise.
2)

$$\begin{aligned}\{f_i\} \text{ Cauchy} &\Rightarrow \{f_i(x)\}, \{\nabla f_i(x)\} \text{ Cauchy} \\ &\Rightarrow \text{ exists } y_x \in \mathbb{R},\ \overrightarrow{y}'_x \in \mathbb{R}^n \text{ such that } f_i(x), \nabla f_i(x) \rightarrow\ y_x, \overrightarrow{y}'_x\end{aligned}$$

3) Def. $f(x) = y_x$, $\overrightarrow{g}(x) = \overrightarrow{y}'_x$, $x \in \Omega$. As in the previous exercise:

$$f \in C(\overline{\Omega}),\ \|f_i - f\|_\infty \to 0$$

$$\overrightarrow{g} \in [C(\overline{\Omega})]^n,\ \|\nabla f_i - \overrightarrow{g}\|_\infty \to 0$$

4) Check: $\nabla f = \overrightarrow{g}$. For any $\epsilon > 0$,

$$\left|\frac{f(x+he_j) - f(x)}{h} - g_j(x)\right| \le |D_{e_j} f(x) - D_{e_j} f_k(x)| + |D_{e_j} f_k(x) - g_j(x)| < \epsilon$$

since the first term on the right-hand side is less than $\frac{\epsilon}{2}$ for k large enough and the second term is less than $\frac{\epsilon}{2}$ for h small enough.

5) Conclude $f_j \to f$ in $C^1(\overline{\Omega})$ and the space is complete.

□

c) Let Ω be a bounded set,

$$\left(C(\overline{\Omega})\ ;\quad < f, g >= \int_\Omega fg\right)$$

is an inner product spcae which is not a Hilbert space.

Proof.

1) $< \cdot, \cdot >$ well defined on $C(\overline{\Omega})$ (if Ω nice)
2) $< \cdot, \cdot >$ inner product on $C(\overline{\Omega})$. Trivial, e.g.

$$< f, f >= 0 \Rightarrow \int |f|^2 dx = 0 \Rightarrow f \equiv 0$$

2) Not complete. Counter-example in $C([-1,1])$. Define

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases} \qquad \text{and} \qquad f_k(x) = \begin{cases} 0 & x < -\frac{1}{k} \\ x + \frac{1}{k} & -\frac{1}{k} \le x < 0 \\ 1 & x \ge 0, \end{cases}$$

and observe that

$$\|f - f_k\|^2 = \int_{-1}^{1} |f - f_k|^2 = \int_{-\frac{1}{k}}^{0} |f - f_k|^2 \le \frac{1}{k} \to \infty$$

since $|f - f_k|^2 \le 1$. Hence $\|f_k - f\| \to 0$ but $f \notin C(\overline{\Omega})$, so $C(\overline{\Omega})$ is not complete.

□

3 (Young's inequality) Let $a, b > 0$, $p > 1$ $q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$a \cdot b \le \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Note that the function $f(x) = e^x$ is convex. Then,

$$\begin{aligned} ab = e^{\ln a + \ln b} &= e^{\frac{1}{p} ln(a^p) + \frac{1}{q} ln(b^q)} \\ &= f(\lambda \ln a^p + (1-\lambda)\ln a^q), \quad \lambda = \frac{1}{p}, \\ &\le \lambda f(\ln a^p) + (1-\lambda) f(\ln a^q), \quad \text{by convexity}, \\ &= \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

□

4 (McOwen 6.1:5) See solutions from last year (2007):
http://www.math.ntnu.no/emner/TMA4305/2007v/Week12.pdf

5 (McOwen 6.1:15)
A bounded bilinear form on a Hilbert space X is a map $B : X \times X \to \mathbb{R}$ such that

(i) $B(ax + by, z) = aB(x, z) + bB(y, z)$
(ii) $B(x, ay + bz) = aB(x, y) + bB(x, z)$
(iii) $|B(x, y)| \le C\|x\|\|y\|$

for all $a, b \in \mathbb{R}$ and $x, y, z \in X$.

Thm 1. *There exists a unique bounded linear operator $A : X \to X$ such that*

$$B(x, y) =< Ax, x >, \quad < \cdot, \cdot > \ \textit{inner product in } X,$$

for all $x, y \in X$.

Proof.

(i) Define $F_x : X \to \mathbb{R}$ as
$$F_x(y) = B(x, y) \quad \text{for all} \quad y \in X.$$
Note that F_x is linear
$$F_x(ay + bz) = aF_x(y) + bF_x(z) \qquad \text{(by (ii))}$$
and bounded
$$|F_x(y)| = |B(x, y)| \le C\|x\|\|y\| \qquad \text{(by (iii))}.$$

(ii) Riesz representation theorem: There exists a unique $z_x \in X$ such that $F_x(y) =< z_x, y >$ for all $y \in X$. Moreover, $\|F_x\| = \|z_x\|$.

(iii) Define $A : X \to X$ by
$$Ax = z_x.$$
A linear:
$$\begin{aligned} < A(ax + by), z > =< z_{ax+by}, z >= F_{ax+by}(z) &= B(ax + by, z) \\ &= aF_x(z) + bF_y(z) \quad \text{by } (i) \\ &=< az_x + bz_y, z > \quad \text{by definition of } z_x \\ (*) \qquad &=< aA_x + bA_y, z > \quad \text{by definition of } A \end{aligned}$$

Take

$$\begin{aligned} z = A(ax + by) - [aA_x + bA_y] \text{ in } (*) \quad &\Rightarrow \quad \|A(ax + by) - [aA_x + bA_y]\| = 0 \\ &\Rightarrow \quad A(ax + by) = aA_x + bA_y \end{aligned}$$

A bounded:

$$\|Ax\| = \|z_x\| = \|F_x\| = \sup_{y\neq 0} \frac{|F_x(y)|}{\|y\|} = \sup_{y\neq 0} \frac{|B(x,y)|}{\|y\|} \leq C\|x\| \quad \text{by } (iii).$$

A unique: Assume $\widetilde{A}$ is such that $B(x,y) =< \widetilde{A}x, y >$ for all $x, y \in X$. Then

$$\begin{aligned}
&0 =< Ax, y > - < \widetilde{A}x, y >=< (A-\widetilde{A})x, y > \\
&\Rightarrow 0 = \|(\widetilde{A}-A)x\| \quad \text{for all } x \in X \quad (\text{take } y = (\widetilde{A}-A)x) \\
&\Rightarrow (\widetilde{A}-A)x = 0 \quad \text{for all } x \in X \\
&\Rightarrow \widetilde{A} = A
\end{aligned}$$

□