Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2008

Solutions Week 17

1 Take $R > 0$ such that $\Omega \subset B_0(R)$ (Ω is bounded) and let $\phi \in C_0^\infty(\Omega)$ be arbitrary. Note that $D^\alpha \phi = 0$ on $\partial\Omega$ for every α. Two integrations by part w.r.t. x_i then give

$$\begin{aligned}\|\phi\|_2^2 &= \int \phi^2 \cdot 1 = -\int \partial_{x_i}(\phi^2)\cdot x_i = -\int 2\phi\phi_{x_i}\cdot x_i \\ &= \int \partial_{x_i}(2\phi\phi_{x_i})\cdot \frac{1}{2}x_i^2 = \int (2\phi_i^2 + 2\phi\phi_{x_i x_i})\cdot\frac{1}{2}x_i^2\end{aligned}$$

Now we sum over i and use that $|x_i| \le |x| \le R$,

$$n\|\phi\|_2^2 = \int (2|\nabla\phi|^2 + 2\phi\Delta\phi)\cdot\frac{1}{2}x_i^2 \le R^2 \int (|\nabla\phi|^2 + |\phi\Delta\phi|).$$

Another integration by parts show that

$$\int |\nabla\phi|^2 = \sum\int \phi_{x_i}^2 = -\sum\int \phi\phi_{x_i x_i}, \tag{1}$$

and the two last inequalities in combination with Cauchy-Schwartz inequality give

$$n\|\phi\|_2^2 \le R^2 2\|\phi\|_2\|\Delta\phi\|_2,$$

or

$$\|\phi\|_2 \le K\|\Delta\phi\|_2, \qquad \text{where} \qquad K = \frac{2\,\mathrm{diam}\,(\Omega)}{n}. \tag{2}$$

The same inequality holds for any element $u \in H_0^(\Omega)$ by density. By definition of $H_0^2(\Omega)$ there is a sequence $\{\phi_i\}_i \subset C_0^\infty(\Omega)$ such that

$$\lim \|u - \phi_i\|_{2,2} = 0,$$

then by the triangle inequality and inequality (2),

$$\|u\|_2 \le \|\phi_i\|_2 + \|u-\phi_i\|_2 \le K\|\Delta\phi_i\|_2 + \|u-\phi_i\|_2 \le K\|\Delta u\|_2 + K\|\Delta\phi_i - \Delta u\|_2 + \|u-\phi_i\|_2.$$

Since this result holds for all i we can send $i \to \infty$ to get

$$\|u\|_2 \le K\|\Delta u\|_2.$$

2 **OBS:** This exercise turned out to be harder that I first thought! It will not be given in this form in the future.

The idea is to prove that $(H_0^2(\Omega), (\cdot,\cdot))$ is a Hilbert space when

$$(u,v) = \int_\Omega \Delta u \Delta v,$$

and that

$$F(v) = \int_\Omega f v$$

is a bounded linear functional on $(H_0^2(\Omega), (\cdot,\cdot))$ when $f \in L^2(\Omega)$.

We can then use Riesz representation theorem to conclude that there is a unique $u \in H_2^2(\Omega)$ such that

$$(u,v) = F(v) \quad \text{for all} \quad v \in H_0^2(\Omega),$$

and hence there is a unique weak solution of (1).

1) $(\cdot,\cdot)$ is an inner product on $H_0^2(\Omega)$: Let $u, v, w \in H_0^2(\Omega)$, $a, b \in \mathbb{R}$, then
   i) $(u,u) \geq 0$
   ii) $0 = (u,u) = \|\Delta u\|_2^2 \underset{\text{exercise 1}}{\Longrightarrow} \|u\|_2 = 0 \Longrightarrow u = 0 \; a.e. \text{ in } \Omega \Longrightarrow u = 0 \text{ in } H_0^2(\Omega).$
   iii) $(au + bv, w) = a(u,w) + b(v,w)$
   iv) $(u,v) = (v,u)$
2) The induced norm $|u|^2 = (u,u)$ is equivalent to the H^2 norm $\|\cdot\|_{2,2}$, and hence $(H_0^2(\Omega), |\cdot|_{2,2})$ is complete since $(H_0^2(\Omega), \|\cdot\|_{2,2})$ is complete:
   It is obvious that
   $$|u|_{2,2} \leq \|u\|_{2,2} \quad \text{for} \quad u \in H_0^2(\Omega).$$
   To prove the opposite inequality, we need the Poincare type inequality from Exercise 1:
   $$\|u\|_2 \leq K\|\Delta u\|_2 \quad \text{for} \quad u \in H_0^2(\Omega),$$
   the (interpolation) inequality that follows from identity (1) in Exercise 1:
   $$\|\nabla u\|_2 \leq \|u\|_2 \|\Delta u\|_2 \quad \text{for} \quad u \in H_0^2(\Omega),$$
   and finally we need an inquality which is not elemtary: There is a constant $C > 0$ such that
   $$\|D^2 u\|_2 \leq C\|\Delta u\|_2 \quad \text{for} \quad u \in H_0^2(\Omega).$$
   We will not prove this inequality - its proof can be found in Stein: "Singular integrals and ..." and requires the theory of singular integral. This proof is not a part of this course.
   By these inequalities it follows that
   $$\|u\|_{2,2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2 + \|D^2 u\|_2^2 \leq (K^2 + K + C^2)\|\Delta u\|_2^2 = (K^2 + K + C^2)|u|_{2,2}^2,$$
   and hence the norms are equivalent.
3) F is a bounded linear functional: This is obvious e.g.
   $$|F(v)| \leq \|f\|_2 \|v\|_2 \leq K\|f\|_2 |v|_{2,2}.$$

Solutions to the rest of the exercises can be found in Solutions Problems given for Week 12, 2007:

http://www.math.ntnu.no/emner/TMA4305/2007v/Week12.pdf