TMA4305 Partial Differential Equations Spring 2008

Solutions for week 18

1 **a)** Consider $g \in H^1(\Omega)$. The set

$$\mathscr{A} = \{u \in H^1(\Omega) : u - g \in H_0^1(\Omega)\}$$

is weakly closed in $H^1(\Omega)$.

Proof. $\mathscr{A}$ weakly closed in $H^1(\Omega)$ means that

$$\mathscr{A} \ni u_k \rightharpoonup u \quad \text{in} \quad H^1(\Omega) \qquad \Rightarrow \qquad u \in \mathscr{A}.$$

Note that

$$\begin{aligned}
&\mathscr{A} \ni u_k \rightharpoonup u \quad \text{in} \quad H^1(\Omega) \\
&\Downarrow \\
&H_0^1(\Omega) \ni (u_k - g) \rightharpoonup (u - g) \quad \text{in} \quad H_0^1(\Omega) \\
&\Downarrow \\
&H_0^1(\Omega) \ni (u_k - g), \quad \|u_k - g\|_{1,2} \le M \quad \text{for all} \quad k \\
&\Downarrow \\
&\text{There exists } \{u_{k_j} - g\} \subset \{u_k - g\}, w \in H_0^1(\Omega) \quad \text{such that} \quad u_{k_j} - g \rightharpoonup w
\end{aligned}$$

But, since $(u_{k_j} - g) \rightharpoonup (u - g)$ and the weak limit is unique,

$$u - g = w \in H_0^1(\Omega) \Rightarrow u \in \mathscr{A}$$

□

b) Define

$$F(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + f u)$$

Consider a function $g \in H^1(\Omega)$ and the set

$$\mathscr{A} = \{u \in H^1(\Omega) : u - g \in H_0^1(\Omega)\}$$

Then,

$$F(v) \ge C_1 \|v\|_{1,2}^2 - C_2$$

for all $v \in \mathscr{A}$ and a $C_1 > 0$.

Proof. Obs 1: For $\epsilon > 0$ and $a, b \in \mathbb{R}$,

$$(\overline{a} - \overline{b}) \ge 0 \quad \Leftrightarrow \quad 2\overline{a}\overline{b} \le \overline{a}^2 + \overline{b}^2 \quad \Rightarrow \quad 2ab \le \frac{a^2}{\epsilon} + \epsilon b^2 \quad \text{when} \quad \overline{a} = \frac{a}{\sqrt{\epsilon}}, \overline{b} = \epsilon b^2.$$

Obs 2: $\|u\| \le \|u - g\| + \|g\|$ and $\|g\| \le \|u - g\| + \|u\|$. So,

$$\|u - g\|^2 \ge (\|u\| - \|g\|)^2 = \|u\|^2 - 2\|u\|\|g\| + \|g\|^2.$$

Hence by Obs 1 with $\epsilon = \frac{1}{2}$,

$$\|u - g\|^2 \ge \frac{1}{2}\|u\|^2 - \|g\|^2$$

Therefore,

$$\begin{aligned}\int |\nabla u|^2 &\geq \frac{1}{2}\int |\nabla(u-g)|^2 - \int |\nabla g|^2 \qquad \text{(Obs 2 with } u+g \text{ instead of } u)\\ &\geq \frac{1}{2}\frac{1}{1+C_\Omega}\|u-g\|_{1,2}^2 - \|g\|_{1,2} \qquad \text{(Poincare: } \|\phi\|_2^2 \leq C_\Omega \|\nabla\phi\|_2^2,\ \phi\in H_0^1(\Omega))\\ &\geq \frac{1}{4}\frac{1}{1+C_\Omega}\|u\|_{1,2}^2 - \Big(\frac{1}{2}\frac{1}{1+C_\Omega}+1\Big)\|g\|_{1,2}^2 \qquad \text{(Obs 2 again).}\end{aligned}$$

Let

$$K = \frac{1}{4}\frac{1}{1+C_\Omega} \qquad \text{and} \qquad \overline{K} = \Big(\frac{1}{2}\frac{1}{1+C_\Omega}+1\Big),$$

and note that from Obs 1 with $\epsilon = \frac{K}{2}$ we get

$$\int |fu| \leq \frac{\|f\|_2^2}{2\epsilon} + \frac{\epsilon}{2}\|u\|_2^2 \leq \frac{\|f\|_2^2}{2\epsilon} + \frac{\epsilon}{2}\|u\|_{1,2}^2 = \frac{K}{4}\|u\|_{1,2} + \frac{\|f\|_2^2}{K}.$$

Therefore, using the previous two relationships on $\int |\nabla u|^2$ and $\int |fu|$ and the fact that $u \in \mathscr{A}$,

$$\begin{aligned}F(u) &\geq \frac{1}{2}\int |\nabla u|^2 - \int |fu|\\ &\geq \frac{1}{2}K\|u\|_{1,2}^2 - \overline{K}\|g\|_{1,2}^2 - \frac{1}{4}K\|u\|_{1,2}^2 - \frac{1}{K}\|f\|_2^2\\ &\geq \frac{1}{4}K\|u\|_{1,2}^2 - (\overline{K}\|g\|_{1,2}^2 + \frac{1}{K}\|f\|_2^2)\end{aligned}$$

□

2 See solutions from last year (2007):
http://www.math.ntnu.no/emner/TMA4305/2007v/Week13.pdf

3 (McOwen 7.1:8 b)
Let $\Omega \subset \mathbb{R}^2$ be bounded. Consider

$$F(u) = \int_\Omega \sqrt{1+u_x^2+u_y^2}\,dxdy$$

where $u \in H^1(\Omega)$.

$$\mathscr{A} = \{u \in H^1(\Omega) : u-g \in H_0^1(\Omega)\} = \{u = g+v : v \in H_0^1(\Omega)\}$$

The function u has a critical point of F on $\mathscr{A}$ if

$$0 = D_v F(u) = \lim_{t\to 0}\frac{F(u+tv)-F(u)}{t}$$

for all $v \in H^1(\Omega)$ such that $u+tv \in \mathscr{A}$ for t small.
Obs: $u+tv \in \mathscr{A}$ for t small implies $u \in H_0^1(\Omega)$.
Let

$$f(p,q) = \sqrt{1+p^2+q^2}$$

and note that $f_p = \frac{p}{f(p,q)}$, $f_q = \frac{q}{f(p,q)}$, and f, f_p, f_q continuous. Formally:

$$\begin{aligned}D_v F(u) &= \frac{d}{dt}F(u+tv)\Big|_{t=0} = \int_\Omega \frac{\partial}{\partial t} f(u_x+tv_x, u_y+tv_y)\Big|_{t=0}\\ &= \int_\Omega \Big(\frac{(u_x+tv_x)v_x}{f(u_x+tv_x,u_y+tv_y)} + \frac{(u_y+tv_y)v_y}{f(u_x+tv_x,u_y+tv_y)}\Big)\Big|_{t=0}\\ &= \int_\Omega \frac{u_xv_x+u_yv_y}{f(u_x,u_y)} = \int_\Omega \frac{\nabla u\cdot\nabla v}{\sqrt{1+|\nabla u|^2}}\end{aligned}$$

These computations are correct if e.g. $u, v \in C^\infty(\overline{\Omega})$ and then they hold by density for $u, v \in H^1(\Omega)$. Hence: $u \in H^1(\Omega)$ critical point of F on $\mathscr{A}$ if the Euler-Lagrange equation holds:

$$0 = D_v(F) = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

Assume $u \in C^2(\Omega)$, integration by parts in the previous expression gives

$$0 = -\int_\Omega \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) v \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

The variational lemma then implies

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = 0$$

Obs:

$$\begin{aligned}
\partial_{x_i}\Big(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\Big) &= \frac{u_{x_i x_i}}{\sqrt{1+|\nabla u|^2}} + u_{x_i}\partial_{x_i}\frac{1}{\sqrt{1+|\nabla u|^2}} \\
&= \frac{u_{x_i x_i}}{\sqrt{1+|\nabla u|^2}} - u_{x_i}\frac{1}{2}\frac{1}{(1+|\nabla u|^2)^{\frac{3}{2}}}\sum 2u_{x_j}u_{x_j x_i} \\
&= \frac{(1+|\nabla u|^2)u_{x_i x_i} - \sum u_{x_i}u_{x_j}u_{x_j x_i}}{(1+|\nabla u|^2)^{\frac{3}{2}}}
\end{aligned}$$

Hence

$$0 = \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \frac{(1+|\nabla u|^2)\Delta u - \sum\sum u_{x_i}u_{x_j}u_{x_j x_i}}{(1+|\nabla u|^2)^{\frac{3}{2}}}$$

So, since we are in $\mathbb{R}^2$,

$$0 = (1+u_x^2+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2+u_y^2)u_{yy}$$

In other words, the abouve Euler Langrance equation is a weak formulation of a minimal surface equation!