Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2008

Problem Set for "Tiltaksuker"

This problem set is based on a problem set given for TMA4305 in 2007 by Sigmund Selberg.

$\boxed{1}$ Solve using the method of characteristics:

(a) $xu_x + yu_y = 2u$, $u(x,1) = g(x)$.

(b) $uu_x + u_y = 1$, $u(x,x) = x/2$.

$\boxed{2}$ Consider Burgers' equation

$$u_t + uu_x = 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty)$$

with initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 0, \\ 2 & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 1. \end{cases}$$

a) Find the solution u also satisfying the *entropy condition*

$$u_l > u_r \quad \text{across any shock.}$$

(This condition ensures uniqueness, and it can be justified on physical grounds. Cf. Section 1.2.b in McOwen, and the Remark at the end of that section.)

b) Draw a picture of the shocks and characteristics in the (x,t)-plane.

$\boxed{3}$ (a) Show that the following equation is hyperbolic:

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0.$$

(b) Transform the equation to canonical coordinates.

(c) Find the general solution $u(x,y)$.

(d) Find a solution that satisfies $u(-x,2x) = x$ and $u(x,0) = \sin 2x$.

$\boxed{4}$ (McOwen 2.3:16) Consider an m-th order differential operator and its principal symbol:

$$Lu = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u \qquad \text{and} \qquad \sigma_L(x;\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \qquad (x,\xi \in \mathbb{R}^n).$$

Prove that L is *elliptic* at x, i.e.

$$\sigma_L(x;\xi) \ne 0 \qquad \text{for all } \xi \in \mathbb{R}^n,\ \xi \ne 0,$$

only when m is an *even* integer. (Hint: Consider $\int_{|\xi|=1} \sigma_L(x;\xi)\, dS_\xi$)

5 The purpose of this exercise is to prove that every linear ordinary differential operator with constant coefficients has a fundamental solution. Let

$$L = \sum_{j=0}^{k} c_j \left(\frac{d}{dx}\right)^j, \qquad c_j = \text{const}, \quad c_k \neq 0.$$

(L is genuinely k-th order). Let v be the solution of

$$Lv = 0, \quad t > 0; \qquad v(0) = \cdots = v^{(k-2)}(0) = 0, \; v^{(k-1)}(0) = c_k^{-1}.$$

(This solution exists by ODE theory.) Prove that

$$F(x) = \begin{cases} v(x) & x > 0 \\ 0 & x < 0 \end{cases}$$

is a fundamental solution of L, i.e. $LF = \delta$.

6 **a)** Show that

$$F(x,y) = 1_{\{x>0,y>0\}}(x,y) = \begin{cases} 1 & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$$

is a fundamental solution for $\partial_x \partial_y$ in $\mathbb{R}^2$.

b) Show that

$$K(x) = -\frac{e^{-c|x|}}{4\pi |x|}$$

is a fundamental solution for $\Delta - c^2$ in $\mathbb{R}^3$.

7 Solve the problem

$$u_{tt} - 4u_{xx} = e^x + \sin t, \qquad u(x,0) = 0, \qquad u_t(x,0) = \frac{1}{1+x^2},$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}$.

8 (a) Show that the general radial solution to the 3d wave equation (with $c = 1$) is

$$u(x,t) = \frac{1}{r}\left[\phi(r+t) + \psi(r-t)\right] \qquad (r = |x|),$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are arbitrary.

(b) Solve the Cauchy problem for the 3d wave equation with radial data:

$$u_{tt} - \Delta u = 0, \qquad u(x,0) = f(|x|), \qquad u_t(x,0) = g(|x|),$$

where f, g are defined on $[0,\infty)$.
(*Hint:* Extend f, g to even functions on $\mathbb{R}$ and find a formula similar to d'Alembert formula)

(c) Let u, f, g be as in part (b). Show that $u(0,t) = f(t) + tf'(t) + tg(t)$.
Thus, u is generally no better than C^k if $f \in C^{k+1}$ and $g \in C^k$.

9 (McOwen 3.2:6) Let u solve

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u = g, \quad u_t = h & \text{on} \quad \mathbb{R}^n \times \{0\}, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^n)$.

a) For $n = 3$, show that
$$|u(x,t)| \le \frac{C}{t} \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty).$$

b) Is a similar result true for $n = 2$?

$\boxed{10}$ Suppose Ω is a bounded domain with smooth boundary, and suppose
$$u \in C^2\big(\Omega \times (0,T)\big) \cap C^1\big(\overline{\Omega} \times (0,T)\big)$$
satisfies
$$\begin{cases} u_t = \Delta u & \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \textit{or} \quad \frac{\partial u}{\partial \nu} = 0 & \text{in} \quad \partial\Omega \times (0,T), \end{cases}$$
Prove that
$$f(t) = \int_\Omega u(x,t)^2\,dx \qquad (0 < t < T),$$
is nonincreasing.

Hint: Show that $u(u_t - \Delta u) = \frac{1}{2}\partial_t\big(u^2\big) - \operatorname{div}(u\nabla u) + |\nabla u|^2$, and integrate over Ω.

$\boxed{11}$ **a)** (McOwen 4.1:8) Hopf Lemma.

Assume:

(i) Ω is a bounded domain in $\mathbb{R}^n$ satisfying an *interior ball condition*:

for every $x \in \partial\Omega$ there exists a ball $B = \{y : |y - y_0| < r\}$ such that $B \subset \Omega$ and $\partial\Omega \cap \bar{B} = \{x\}$

(ii) $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy
$$\Delta u \ge 0 \quad \text{in} \quad \Omega.$$

(iii) There is an $x_0 \in \partial\Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$.

Prove that either
$$\frac{\partial u}{\partial \nu}(x_0) > 0 \qquad \text{or} \qquad u \equiv \text{constant} \quad \text{in} \quad \bar{\Omega},$$
where ν denote the unit exterior normal of $\partial\Omega$ and $\frac{\partial u}{\partial \nu}(x) = \nu(x) \cdot \nabla u(x)$ for $x \in \partial\Omega$.

b) Use part a) to prove the *strong maximum principle*:

If (i) and (ii) hold, then either $\quad u(x) < \max_{\bar{\Omega}} u$ for all $x \in \Omega \quad$ or $\quad u \equiv$constant in $\bar{\Omega}$.

c) (Uniqueness results for the Robin and Neumann problem)

Let $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be two solutions of
$$\begin{cases} \Delta u = f(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} + \alpha(x) u = h(x) & \text{on} \quad \partial\Omega, \end{cases}$$
where $\alpha \ge 0$, f, α, h are continuous, and Ω satisfy (i).

Use part a) and b) to prove that

(1) $\alpha \not\equiv 0$ (Robin case) $\Rightarrow u \equiv v \quad$ in $\quad \bar{\Omega}$.

(2) $\alpha \equiv 0$ (Neumann case) $\Rightarrow u - v \equiv$ constant $\quad$ in $\quad \bar{\Omega}$.

$\boxed{12}$ Let A, B, C, and R be real $n \times n$-matrices.

We say that $A = (a_{ij})$ is *positive definite* (resp. *positive semi-definite* if
$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j > 0 \quad (\text{resp. } \ge 0) \quad \text{for all } \xi \in \mathbb{R}^n,\ \xi \ne 0.$$
(And A is *negative (semi) definite* if $-A$ is positive (semi) definite).

(a) Show that if A is positive semi-definite, then $a_{ii} \geq 0$ for $i = 1, \ldots, n$.
Moreover, if λ is an eigenvalue of A, then $\lambda \geq 0$.

(b) Prove that if A and B are symmetric and positive semi-definite, then $\operatorname{tr}(AB) \geq 0$, where tr denotes the trace. (*Hint:* Diagonalize A using an orthonormal basis of eigenvectors. Use part (a) and the fact that $\operatorname{tr}(R^t C R) = \operatorname{tr}(C)$ for all C if R is an orthogonal matrix.)

13 Let $u : \Omega \to \mathbb{R}$ be C^2. Prove that if u has a local maximum at at point $x_0 \in \Omega$, then the symmetric $n \times n$-matrix $D^2 u(x_0)$ with entries $\partial_i \partial_j u(x_0)$ is negative semi-definite.

(*Hint:* Given $\xi \in \mathbb{R}^n$, $\xi \neq 0$, define $\phi(t) = u(x_0 + t\xi)$ for t in a small interval around 0.)

14 The purpose of this exercise is to prove the weak maximum principle (cf. (16) in Section 4.1 of McOwen) for a more general elliptic operator than the Laplace operator.

Let Ω be a bounded domain in $\mathbb{R}^n$, and let

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i \partial_j + \sum_{i=1}^{n} b_i(x)\partial_i,$$

where a_{jk} and b_j are continuous functions on $\overline{\Omega}$ and the matrix (a_{ij}) is symmetric (so $a_{jk} = a_{kj}$) and positive definite, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j > 0 \qquad \text{for all } x \in \overline{\Omega} \text{ and all } \xi \in \mathbb{R}^n \text{ with } \xi \neq 0. \tag{1}$$

(The operator L is elliptic and (1) is called the *ellipticity condition.*)

(a) Show that if $v \in C^2(\Omega)$ satisfies $Lv > 0$ in Ω, then v cannot have a local maximum in Ω. (*Hint:* Use the two previous problems to get a contradiction if we assume that a local maximum exists.)

(b) Show that if $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and $M > 0$ is sufficiently large, then $w(x) = \exp(-M|x - x_0|^2)$ satisfies $Lw > 0$ in Ω.

(c) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and that $Lu = 0$ in Ω. Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(*Hint:* Show that this conclusion holds for $v = u + \varepsilon w$, where w is as above and $\varepsilon > 0$.)