Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions for Problem set for "tiltaksuker"

1

a) We solve

$$xu_x + yu_y = 2u, \qquad u(x,1) = g(x),$$

by the method of characteristics. The ODEs for the characteristics are:

$$\frac{dx}{dt} = x\ , \qquad x(0) = s,$$
$$\frac{dy}{dt} = y\ , \qquad y(0) = 1,$$
$$\frac{dz}{dt} = 2z, \qquad z(0) = g(s).$$

We solve these, obtaining

$$x = se^t, \qquad y = e^t, \qquad z = g(s)e^{2t},$$

which gives $s = x/y$ and $e^{2t} = y^2$, hence

$$u(x,y) = z = g\left(\frac{x}{y}\right)y^2.$$

NB! Always check your answer by plugging it into the equation/initial condition.

b) Now consider

$$uu_x + u_y = 1, \qquad u(x,x) = x/2.$$

The ODEs for the characteristics are:

$$\frac{dx}{dt} = z, \qquad x(0) = s,$$
$$\frac{dy}{dt} = 1, \qquad y(0) = s,$$
$$\frac{dz}{dt} = 1, \qquad z(0) = s/2.$$

We solve (first for y and z, then x), obtaining

$$x = \frac{t^2 + st}{2} + s, \qquad y = t + s, \qquad z = t + \frac{s}{2}.$$

Now we need to express z in terms of x and y only, so we need to express s and t as functions of x and y. Let us observe that

$$x = \frac{t}{2}(t+s) + s,$$

so if we add t, we get

$$x + t = \left(\frac{t}{2} + 1\right)(t+s) = \left(\frac{t}{2} + 1\right)y,$$

hence

$$t = \frac{y-x}{1-y/2} = \frac{2x-2y}{y-2}.$$

Combining this with $y = t + s$ gives

$$s = y - t = y - \frac{2x - 2y}{y - 2} = \frac{y^2 - 2x}{y - 2},$$

and we conclude that

$$\underline{\underline{u(x,y) = z = \frac{2x - 2y}{y - 2} + \frac{y^2/2 - x}{y - 2} = \frac{y}{2} + \frac{x - y}{y - 2}.}}$$

$\boxed{2}$ We are going to solve Burgers' equation

$$u_t + uu_x = 0$$

for $t > 0$, with initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 0, \\ 2 & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Moreover, we require that u satisfy the condition $u_l > u_r$ across a shock.

It is important to draw a sketch of the initial function, call it $h(x)$, and of the initial shocks and the characteristic curves. Note that for Burgers' equation, the characteristics are straight lines of the form

$$x = x_0 + tu_0, \tag{1}$$

where $u_0 = u(x_0, 0)$ denotes the constant value of u along the characteristic.

First look at $x = -1$, where h jumps from value $u_l = 1$ (to the left of $x = -1$) to value $u_r = 0$ (to the right of $x = -1$). Therefore, a shock will emanate from this point, described by a curve $x = \xi_1(t)$, $\xi_1(0) = -1$. By Rankine-Hugoniot (R-H) we get

$$\xi_1'(t) = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{u_r + u_l}{2} = \frac{1}{2},$$

hence $\xi_1(t) = -1 + t/2$. To the left of this shock, u equals u_l, and to the right u equals u_r.

Then we look $x = 0$ where h jumps up from 0 to 2. Here we get a rarefaction wave (not a shock, in view of the entropy condition!). To find the rarefaction solution we just use (1) with $x_0 = 0$, which gives $u = x/t$ in the wedge between $x = 0$ and $x = 2t$.

Finally, at $x = 1$, h jumps down from 2 to 0. Here we get a shock $x = \xi_2(t)$ emanating from $x = 1$. R-H gives $\xi_2'(t) = \frac{1}{2}(2 + 0) = 1$, hence $\xi_2(t) = 1 + t$.

Let us summarize (remember to sketch!):

$$u(x,t) = \begin{cases} 1 & \text{if } x < -1 + t/2, \\ 0 & \text{if } -1 + t/2 < x \le 0, \\ x/t & \text{if } 0 < x < 2t, \\ 2 & \text{if } 2t \le x < 1 + t, \\ 0 & \text{if } x > 1 + t. \end{cases} \tag{2}$$

This solution is valid up to time $t = 1$, where the shock $x = 1 + t$ collides with the right edge $x = 2t$ of the rarefaction wedge. This creates a new shock, emanating from $(x,t) = (2,1)$. The shock curve $x = \xi_4(t)$, with $\xi_4(1) = 2$, is found using R-H:

$$\xi_4'(t) = \frac{u_r + u_l}{2} = \frac{1}{2}u_l = \frac{\xi_4(t)}{2t},$$

where we used that $u = x/t$ to the left of the shock, and $u = 0$ to the right of the shock, by (2). Thus, we have a separable ODE for ξ_4. Solving this, we find $\xi_4(t) = 2t^{1/2}$. Hence, for $t > 1$,

$$u(x,t) = \begin{cases} 1 & \text{if } x < -1 + t/2, \\ 0 & \text{if } -1 + t/2 < x \le 0, \\ x/t & \text{if } 0 < x < 2t^{1/2}, \\ 0 & \text{if } x > 2t^{1/2}. \end{cases} \tag{3}$$

This aolution is valid until $t = 2$, where the shock $x = -1 + t/2$ hits the left edge $x = 0$ of the rarefaction wedge. This creates a new shock $x = \xi_5(t)$, with $\xi_5(2) = 0$. Since $u_l = 1$ and $u_r = x/t$, we get from R-H:

$$\xi_5'(t) = \frac{u_r + u_l}{2} = \frac{1}{2} + \frac{\xi_5(t)}{2t}.$$

This we can solve using the integrating factor $e^{-(1/2)\log t} = t^{-1/2}$: $\xi_5(t) = t - (2t)^{1/2}$. Hence, for $t > 2$,

$$u(x,t) = \begin{cases} 1 & \text{if } x < t - (2t)^{1/2}, \\ x/t & \text{if } t - (2t)^{1/2} < x < 2t^{1/2}, \\ 0 & \text{if } x > 2t^{1/2}. \end{cases} \tag{4}$$

The two shock curves $x = t - (2t)^{1/2}$ and $x = 2t^{1/2}$ collide when $t = (2+\sqrt{2})^2$ [then $x = 2(2+\sqrt{2})$], at which point a new shock develops. Since $u_l = 1$ and $u_r = 0$, by (3), we see from R-H that the shock has speed $1/2$, hence it is described by

$$x = 2(2+\sqrt{2}) + \frac{1}{2}[t - (2+\sqrt{2})^2] = \frac{t}{2} + \frac{1}{2}(2+\sqrt{2})(2-\sqrt{2}) = \frac{t}{2} + 1.$$

So for $t > (2+\sqrt{2})^2$,

$$u(x,t) = \begin{cases} 1 & \text{if } x < 1 + t/2, \\ 0 & \text{if } x > 1 + t/2. \end{cases} \tag{5}$$

Together, (2), (3), (4) and (5) describe the solution for all $t > 0$! Figure 1 shows a sketch of the xt-plane, showing the shocks and the rarefaction wedge.

3 **a)** We are asked to show that the following equation is hyperbolic:

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0. \tag{6}$$

This is of the form $au_{xx} + bu_{xy} + cu_{yy} = 0$, with $a = 1$, $b = 6$ and $c = -16$. The condition for hyperbolicity is $b^2 - 4ac > 0$. Here $b^2 - 4ac = 36 + 64 = 100 > 0$, so the equation is hyperbolic.

b) We transform the equation to canonical coordinates. To find the characteristic curves, in the form of graphs $y = f(x)$, we solve (see p. 50 of McOwen)

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm 10}{2} = \begin{cases} -2, \\ 8, \end{cases}$$

which gives $y = y_0 - 2x$ and $y = y_0 + 8x$. So the characteristic curves are $y + 2x = \text{const}$ and $y - 8x = \text{const}$. The canonical coordinates are therefore

$$\mu = y + 2x, \qquad \eta = y - 8x. \tag{7}$$

The chain rule gives

$$\begin{aligned} u_x &= u_\mu \mu_x + u_\eta \eta_x = 2u_\mu - 8u_\eta, \\ u_y &= u_\mu \mu_y + u_\eta \eta_y = u_\mu + u_\eta, \\ u_{xx} &= 2u_{\mu\mu}\mu_x + 2u_{\mu\eta}\eta_x - 8u_{\mu\eta}\mu_x - 8u_{\eta\eta}\eta_x = 4u_{\mu\mu} - 32u_{\mu\eta} + 64u_{\eta\eta}, \\ u_{xy} &= 2u_{\mu\mu}\mu_y + 2u_{\mu\eta}\eta_y - 8u_{\mu\eta}\mu_y - 8u_{\eta\eta}\eta_y = 2u_{\mu\mu} - 6u_{\mu\eta} - 8u_{\eta\eta}, \\ u_{yy} &= u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_{\mu\eta}\mu_y + u_{\eta\eta}\eta_y = u_{\mu\mu} + 2u_{\mu\eta} + u_{\eta\eta}. \end{aligned}$$

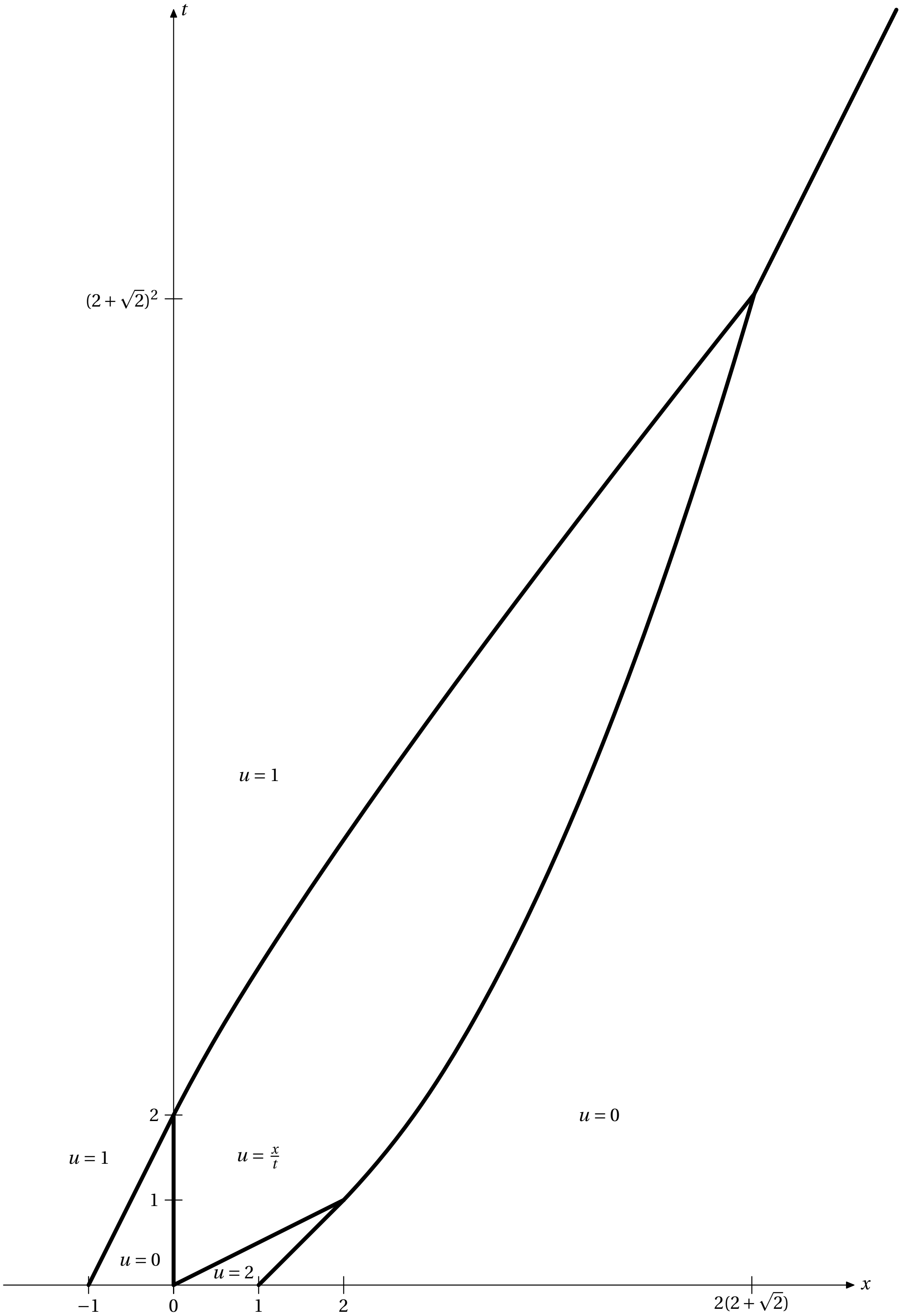

Figure 1: The solution of exercise 2.

Plugging this into (6), we get

$$4u_{\mu\mu} - 32u_{\mu\eta} + 64u_{\eta\eta} + 6\left(2u_{\mu\mu} - 6u_{\mu\eta} - 8u_{\eta\eta}\right) - 16\left(u_{\mu\mu} + 2u_{\mu\eta} + u_{\eta\eta}\right) = 0,$$

which simplifies to $-100u_{\mu\eta} = 0$, i.e.,

$$u_{\mu\eta} = 0. \tag{8}$$

c) We find the general solution $u(x,y)$. The general solution of (8) is

$$u(\mu,\eta) = F(\mu) + G(\eta),$$

for arbitrary functions F and G. Substituting (7) into this gives

$$\underline{\underline{u(x,y) = F(y+2x) + G(y-8x).}}$$

d) Here we are asked to find a solution that satisfies $u(-x,2x) = x$ and $u(x,0) = \sin 2x$. From part (c) we get the equations

$$\begin{aligned} u(-x,2x) &= F(0) + G(10x) = x, \\ u(x,0) &= F(2x) + G(-8x) = \sin 2x. \end{aligned}$$

Thus, $G(s) = s/10 - C$, where C is a constant, and $F(s) = \sin s - G(-4s) = \sin s + 4s/10 + C$. This gives $u(x,y) = \sin(y+2x) + \frac{4}{10}(y+2x) + \frac{1}{10}(y-8x)$, which simplifies to

$$\underline{\underline{u(x,y) = \sin(y+2x) + \frac{y}{2}.}}$$

$\boxed{4}$ (Exercise 2.3.16 from McOwen.) Here we consider an m-th order operator

$$Lu = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u.$$

The principal symbol is

$$\sigma(x;\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \qquad (x,\xi\in\mathbb{R}^n).$$

Now we assume that for a given x, L is *elliptic*, i.e.,

$$\sigma(x;\xi) \neq 0 \qquad \text{for all } \xi\in\mathbb{R}^n,\ \xi\neq 0.$$

We are asked to prove that this implies that m must be an *even* integer. Since x is fixed, let us set $c_\alpha = a_\alpha(x)$ and write $\sigma(\xi) = \sum_{|\alpha|=m} c_\alpha\xi^\alpha$, to simplify the notation. Let us also restrict ξ to lie on the unit sphere, so $|\xi| = 1$. Since $\sigma(\xi)$ is a continuous function on the unit sphere, and it does not vanish at any point on the unit sphere, then it must have a definite sign there. Without loss of generality, we may therefore assume

$$\sigma(\xi) > 0 \qquad \text{for all } |\xi| = 1.$$

But this implies that

$$I := \int_{|\xi|=1} \sigma(\xi)\, dS(\xi) > 0.$$

On the other hand,

$$I = \sum_{|\alpha|=m} c_\alpha I_\alpha,$$

where

$$I_\alpha = \int_{|\xi|=1} \xi^\alpha\, dS(\xi).$$

However, it is a general fact that if we integrate any monomial ξ^α of odd order (i.e., if $|\alpha|$ is odd) over the unit sphere, then we get zero! Therefore, if m were odd, we would have $I = 0$, which is a contradiction. Therefore, m must be even.

It is a good exercise to prove the property of odd-powered monomials stated above. For example, taking $n = 3$, and using coordinates x, y, z on the unit sphere, we consider $I_{pqr} = \int_{S^2} x^p y^q z^r \, dS$, where $p, q, r \geq 0$ are integers and $p+q+r$ is odd. Then at least one of p, q, r must be odd. Without loss of generality, assume p is odd. Then by symmetry (change variables $x \to -x$; this leaves dS unchanged!), we see that $I_{pqr} = -I_{pqr}$, hence $I_{pqr} = 0$.

5 We consider a linear k-th order ordinary differential operator with constant coefficients,

$$L = \sum_{j=0}^{k} c_j \left(\frac{d}{dx} \right)^j .$$

Here the c_j are constants, and we assume $c_k \neq 0$ (so L is genuinely k-th order).

Let v be the solution of $Lv = 0$ with $v(0) = \cdots = v^{(k-2)}(0) = 0$ and $v^{(k-1)}(0) = c_k^{-1}$. (This solution exists, by ODE theory.)

Now define $F(x) = v(x)$ for $x > 0$ and $F(x) = 0$ for $x < 0$. We are asked to prove that (in the sense of distributions)

$$LF = \delta, \tag{9}$$

i.e., F is a fundamental solution.

By the definition of distributional derivatives, (9) is equivalent to

$$\sum_{j=0}^{k} c_j (-1)^j \left\langle F, \phi^{(j)} \right\rangle = \phi(0) \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}). \tag{10}$$

But by the definition of F,

$$\left\langle F, \phi^{(j)} \right\rangle = \int_{-\infty}^{\infty} F(x)\phi^{(j)}(x)\, dx = \int_0^\infty v(x)\phi^{(j)}(x)\, dx. \tag{11}$$

Using integration by parts, the compact support of ϕ, and the properties of v, we find that for $j = 1, 2, \cdots, k-1$,

$$\begin{aligned} \int_0^\infty v(x)\phi^{(j)}(x)\, dx &= -\underbrace{v(0)\phi^{(j-1)}(0)}_{=0} - \int_0^\infty v'(x)\phi^{(j-1)}(x)\, dx \\ &= (-1)^2 \underbrace{v'(0)\phi^{(j-2)}(0)}_{=0} + (-1)^2 \int_0^\infty v''(x)\phi^{(j-2)}(x)\, dx \\ &= \cdots \\ &= (-1)^j \underbrace{v^{(j-1)}(0)\phi(0)}_{=0} + (-1)^j \int_0^\infty v^{(j)}(x)\phi(x)\, dx, \end{aligned} \tag{12}$$

whereas for $j = k$, the same computation applies, but the last boundary does not vanish:

$$\int_0^\infty v(x)\phi^{(k)}(x)\, dx = (-1)^k \underbrace{v^{(k-1)}(0)\phi(0)}_{=c_k^{-1}\phi(0)} + (-1)^k \int_0^\infty v^{(k)}(x)\phi(x)\, dx. \tag{13}$$

Combining (11)–(13), we see that

$$\sum_{j=0}^{k} c_j (-1)^j \left\langle F, \phi^{(j)} \right\rangle = \sum_{j=0}^{k} c_j \int_0^\infty v^{(j)}(x)\phi(x)\, dx + \phi(0) = \int_0^\infty \underbrace{Lv(x)}_{=0}\phi(x)\, dx + \phi(0) = \phi(0),$$

which proves (10).

6 **a)** Here we are supposed to show that the characteristic function of the first quadrant in the (x, y)-plane (i.e., $F(x, y) = 1$ if $x, y > 0$ and $F(x, y) = 0$ otherwise) is a fundamental solution for $\partial_x \partial_y$ in $\mathbb{R}^2$.

So we have to prove $\partial_x \partial_y F = \delta$, in the sense of distributions on $\mathbb{R}^2$. This is equivalent to saying that

$$\iint_{\mathbb{R}^2} F(x,y)\phi_{xy}(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty \phi_{xy}(x,y)\,dx\,dy = \phi(0,0) \qquad \text{for all } \phi \in C_0^\infty(R^2). \tag{14}$$

But for all y,

$$\int_0^\infty \phi_{xy}(x,y)\,dx = \lim_{N\to\infty} \left[\phi_y(x,y)\right]_{x=0}^{x=N} = -\phi_y(0,y),$$

where we used the compact support. Integrating this in y, we get

$$\int_0^\infty \int_0^\infty \phi_{xy}(x,y)\,dx\,dy = -\int_0^\infty \phi_y(0,y)\,dy = -\lim_{N\to\infty} \left[\phi(0,y)\right]_{y=0}^{y=N} = \phi(0,0),$$

where again we used the compact support. This proves (14).

b) Here we are going to show that $K(x) = -\dfrac{e^{-c|x|}}{4\pi|x|}$ is a fundamental solution for $\Delta - c^2$ on $\mathbb{R}^3$.

First write $K = fg$ for $f(x) = -\dfrac{1}{4\pi|x|}$ and $g(x) = e^{-c|x|}$. For $x \neq 0$, we can use the identity

$$\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g. \tag{15}$$

Straightforward calculations (chain rule etc.) give

$$\nabla f(x) = \frac{1}{4\pi|x|^2}\frac{x}{|x|}, \qquad \Delta f(x) = 0, \qquad \nabla g(x) = -c\frac{x}{|x|}e^{-c|x|}, \qquad \Delta g(x) = c^2 e^{-c|x|} - \frac{2c}{|x|}e^{-c|x|}.$$

Plugging this into (15) gives

$$\Delta K(x) = c^2 K(x) \qquad \text{for } x \neq 0. \tag{16}$$

Let us also note for later use that

$$\nabla K(x) = f(x)\nabla g(x) + g(x)\nabla f(x) = \frac{c}{4\pi|x|}\frac{x}{|x|}e^{-c|x|} + \frac{1}{4\pi|x|^2}\frac{x}{|x|}e^{-c|x|}. \tag{17}$$

Now we need to prove that $\Delta K = c^2 K + \delta$ in $\mathscr{D}'$, or equivalently,

$$\int_{\mathbb{R}^3} K(x)\Delta\phi(x)\,dx = c^2 \int_{\mathbb{R}^3} K(x)\phi(x)\,dx + \phi(0) \quad \text{for every test function } \phi \in C_0^\infty(\mathbb{R}^3). \tag{18}$$

We fix ϕ, and choose M so large that $\left|\phi(x)\right|, \left|\nabla\phi(x)\right|, \left|\Delta\phi(x)\right| \le M$ for all x.
We apply the usual trick of cutting out a small ball $B_\varepsilon(0)$, $\varepsilon > 0$, to the left hand side of (18):

$$\int_{\mathbb{R}^3} K(x)\Delta\phi(x)\,dx = \int_{|x|\le\varepsilon} K(x)\Delta\phi(x)\,dx + \int_{|x|>\varepsilon} K(x)\Delta\phi(x)\,dx =: I + J.$$

Let ν denote the outward pointing unit normal on the surface $|x| = \varepsilon$, so $\nu = x/|x|$. Then by Green's second identity (see p. 107),

$$J = \int_{|x|>\varepsilon} \phi(x)\Delta K(x)\,dx - \int_{|x|=\varepsilon} K\frac{\partial\phi}{\partial\nu}\,dS + \int_{|x|=\varepsilon} \phi\frac{\partial K}{\partial\nu}\,dS =: J_1 + J_2 + J_3. \tag{19}$$

By (18),

$$J_1 = c^2 \int_{|x|>\varepsilon} \phi(x)K(x)\,dx. \tag{20}$$

We estimate

(21) $$|J_2| \le M\frac{e^{|c|\varepsilon}}{4\pi\varepsilon}\int_{|x|=\varepsilon} 1\,dS = Me^{|c|\varepsilon}\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0.$$

The term J_3 can be written, using (17) and the fact that $\nu = x/|x|$:

$$J_3 = \int_{|x|=\varepsilon} \phi\nabla K\cdot\nu\,dS = \frac{ce^{-c\varepsilon}}{4\pi\varepsilon}\int_{|x|=\varepsilon}\phi\,dS + \frac{e^{-c\varepsilon}}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}\phi\,dS.$$

By the mean value theorem for integrals, $\int_{|x|=\varepsilon}\phi\,dS = 4\pi\varepsilon^2\phi(x^*)$ for some x^* with $|x^*| = \varepsilon$, so we conclude, letting $\varepsilon \to 0$, that

(22) $$\lim_{\varepsilon\to 0} J_3 = \phi(0).$$

For the term I we estimate

(23) $$|I| \le Me^{|c|\varepsilon}\int_{|x|\le\varepsilon}\frac{1}{4\pi|x|}\,dx = Me^{|c|\varepsilon}\frac{\varepsilon^2}{2} \to 0 \qquad \text{as } \varepsilon \to 0,$$

where we used spherical coordinates to calculate the integral $\int_{|x|\le\varepsilon}\frac{1}{4\pi|x|}\,dx$. In the same way we obtain

(24) $$\left|c^2\int_{|x|\le\varepsilon} K(x)\phi(x)\,dx\right| \to 0 \qquad \text{as } \varepsilon \to 0.$$

Combining (20), (21), (22), (23) and (24), we conclude that

$$\int_{\mathbb{R}^3} K(x)\Delta\phi(x)\,dx = c^2\int_{\mathbb{R}^3} K(x)\phi(x)\,dx + \phi(0) + o(\varepsilon),$$

where $o(\varepsilon)$ denotes a term which tends to zero as ε tends to zero. We therefore obtain (18).

$\boxed{7}$ We are asked to solve the problem

$$u_{tt} - 4u_{xx} = e^x + \sin t, \qquad u(x,0) = 0, \qquad u_t(x,0) = \frac{1}{1+x^2},$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}$. By D’Alembert’s formula (p. 75), and Duhamel’s formula (Eq. (19) on p. 81),

$$u(x,t) = I + J,$$

where (observe that $c = 2$ in this case)

$$I = \frac{1}{4}\int_{x-2t}^{x+2t}\frac{dy}{1+y^2},$$
$$J = \frac{1}{4}\int_0^t\left(\int_{x-2(t-s)}^{x+2(t-s)}\left(e^y + \sin s\right)dy\right)ds.$$

We solve:

$$I = \frac{1}{4}\arctan(x+2t) - \frac{1}{4}\arctan(x-2t),$$

and

$$\begin{aligned}
J &= \frac{1}{4}\int_0^t\left(e^{x+2(t-s)} - e^{x-2(t-s)} + 4(t-s)\sin s\right)ds\\
&= \frac{1}{4}\left[-\frac{1}{2}e^{x+2(t-s)} - \frac{1}{2}e^{x-2(t-s)} - 4t\cos s - 4\sin s + 4s\cos s\right]_{s=0}^{s=t}\\
&= \frac{1}{4}\left(-e^x - 4\sin t + \frac{1}{2}e^{x+2t} + \frac{1}{2}e^{x-2t} + 4t\right).
\end{aligned}$$

Therefore,

$$\underline{\underline{u(x,t) = \frac{1}{4}\left(\arctan(x+2t) - \arctan(x-2t) + \frac{1}{2}e^{x+2t} + \frac{1}{2}e^{x-2t} - e^x - 4\sin t + 4t\right).}}$$

8 **a)** We are asked to show that the general radial solution to the 3d wave equation (with $c = 1$) is

$$u(x,t) = \frac{1}{r}\left[\phi(r+t) + \psi(r-t)\right] \qquad (r = |x|), \tag{25}$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are arbitrary. Recall that for a radial function $f(x) = f(r)$ (slight abuse of notation here, but no confusion should arise), where $r = |x|$ and $x \in \mathbb{R}^3$, the Laplace operator becomes $\Delta f(x) = f''(r) + (2/r)f'(r)$, for $r > 0$. Therefore, if $u(x,t)$ is radial in x, i.e., $u = u(r,t)$, then the wave equation

$$u_{tt} - \Delta u = 0 \tag{26}$$

becomes

$$u_{tt} - u_{rr} - \frac{2}{r}u_r = 0. \tag{27}$$

(Let us remark that (27) only makes sense for $r > 0$, so it only implies that (26) holds for $x \neq 0$, but if u is C^2 as a function of $x \in \mathbb{R}^3$, then it follows by continuity that (26) holds also for $x = 0$. Therefore, (26) and (27) really are equivalent, if we know that u is C^2.)
The trick one uses to solve (27) is to introduce $v = ru$. Then $v_r = u + ru_r$ and $v_{rr} = 2u_r + ru_{rr}$, hence (27) is equivalent to (multiply both sides of (27) by r)

$$v_{tt} - v_{rr} = 0, \tag{28}$$

which is the 1d wave equation in the variables r and t. As we know, the general solution of this is

$$v(r,t) = \phi(r+t) + \psi(r-t),$$

where ϕ and ψ are arbitrary functions from $\mathbb{R}$ into $\mathbb{R}$. Dividing by r, we then get (25).

b) We are supposed to solve the Cauchy problem for the 3d wave equation with radial data:

$$u_{tt} - \Delta u = 0, \qquad u(x,0) = f(|x|), \qquad u_t(x,0) = g(|x|), \tag{29}$$

where f, g are defined on $[0,\infty)$. We extend f and g to even functions on $\mathbb{R}$, so for $r \geq 0$, we set $f(-r) = f(r)$ and $g(-r) = g(r)$.
By part (a), we know the solution must be of the form (27). The initial conditions give

$$\phi(r) + \psi(r) = rf(r), \tag{30}$$

$$\phi'(r) - \psi'(r) = rg(r). \tag{31}$$

Integrating (31), we get

$$\phi(r) - \psi(r) = \int_0^r sg(s)\,ds. \tag{32}$$

Adding or subtracting (30) and (32) gives

$$\phi(r) = \frac{1}{2}\left(rf(r) + \int_0^r sg(s)\,ds\right), \qquad \psi(r) = \frac{1}{2}\left(rf(r) - \int_0^r sg(s)\,ds\right).$$

Plugging this into (27), we obtain

$$u(x,t) = \frac{1}{2r}\left[(r+t)f(r+t) + (r-t)f(r-t)\right] + \frac{1}{2r}\left(\int_0^{r+t} sg(s)\,ds - \int_0^{r-t} sg(s)\,ds\right) \tag{33}$$

and since

$$\begin{aligned}
-\int_0^{r-t} sg(s)\,ds &= \int_{r-t}^0 sg(s)\,ds \\
&= \int_{-(r-t)}^0 (-y)g(-y)\,(-1)\,dy && \text{change variables to } y = -s,\ dy = -ds \\
&= \int_{t-r}^0 yg(y)\,dy, && \text{since } g(-y) = g(y)
\end{aligned}$$

we get

$$u(x,t) = \underline{\underline{\frac{1}{2r}\left[(r+t)f(r+t)+(r-t)f(r-t)\right] + \frac{1}{2r}\int_{t-r}^{t+r} sg(s)\,ds,}} \tag{34}$$

which is the solution of (29). This is obviously C^2 for $x \neq 0$ (i.e., $r > 0$) if the even extensions of f, g are C^2 on $\mathbb{R}$; what happens at $r = 0$ is less clear, but this is investigated in part (c).

c) We are asked to find $u(0,t)$. Thus, we must take the limit as $r \to 0$ in (34). The first term in (34) we rewrite as

$$\frac{(t+r)f(t+r)-(t-r)f(t-r)}{2r}$$

using the fact that $f(-r) = f(r)$, by the even extension. Clearly,

$$\lim_{r\to 0}\frac{(t+r)f(t+r)-(t-r)f(t-r)}{2r} = \frac{d}{dt}\left[tf(t)\right] = f(t)+tf'(t).$$

Since $sg(s)$ is a continuous function, the mean value theorem for integrals tells us that the second term in (34) can be written as

$$s^* g(s^*) \qquad \text{for some } s^* = s^*(r) \text{ in } (t-r, t+r),$$

hence in the limit $r \to 0$ we get $tg(t)$. Thus, we have shown that

$$u(0,t) = f(t) + tf'(t) + tg(t).$$

This shows that u is generally no better than C^k for $t > 0$ if $f \in C^{k+1}$ and $g \in C^k$! (So going from $t = 0$ to $t > 0$ we immediately “lose” one derivative; note that for the heat equation the situation is quite different, in that non-smooth initial data are immediately smoothed out!)

$\boxed{9}$ **a)** Suppose first $g = 0$ and $h \in C_0^\infty$.
Note that h is bounded (say $|h| \le M$), and $= 0$ for $|x| > R$ and R large enough. By Kirchhoff’s formula,

$$u(x,t) = \frac{t}{4\pi}\int_{|y|=1} h(x+cty)\,dS_y.$$

Changing variables to $z = x + cty$, we have $dS_z = (ct)^2 dS_y$, we find

$$u(x,t) = \frac{t}{4\pi}\int_{|z-x|=ct} h(z)\frac{dS_z}{(ct)^2},$$

hence, for $t > 0$, using the assumptions on h above,

$$|u(x,t)| \le \frac{1}{4\pi c^2 t}\int_{\{|z-x|=ct\}\cap\{|x|\le R\}} M\,dS_y = \frac{M}{4\pi c^2 t}\text{Area}\Big(\big\{|z-x|=ct\big\}\cap\big\{|x|\le R\big\}\Big),$$

But $\{|z-x| = ct\} \cap \{|x| \le R\}$ is a sphere “cap” whose area is bounded independently of x and t (no larger than AR^2 for some absolute constant A as can be calculated explicitly calculated in spherical coordinates). Therefore,

$$|u(x,t)| \le \frac{C}{t},$$

where C is independent of x and t.
This shows the result for $g = 0$. If instead we assume $h = 0$, then by Kirchhoff s formula,

$$\begin{aligned}
u(x,t) &= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|y|=1} g(x+cty)\,dS_y\right)\\
&= \frac{1}{4\pi}\int_{|y|=1} g(x+cty)\,dS_y + \frac{t}{4\pi}\int_{|y|=1} (\nabla g)(x+cty)\cdot(cy)\,dS_y,
\end{aligned}$$

and both terms are covered by the preceding analysis, if $g \in C_0^\infty(\mathbb{R}^3)$.

b) The result fails in 2d. Instead, one can then show that (if $g, h \in C_0^\infty$)

$$|u(x,t)| \le \frac{C}{\sqrt{t}},$$

but this is a little more tricky, and we leave it. Intuitively, the decay as $t \to \infty$ in 3d (part (a)) is due to the fact that the wave spreads out in space, on spheres expanding (with speed c) as time increases, hence the amplitude of the wave attenuates. In 2d the wave has one less direction to spread out in, therefore it will not attenuate as quickly. In 1d, there is no attenutation at all as $t \to \infty$, as is clear from d'Alembert's formula.

$\boxed{10}$ Suppose Ω is a bounded domain with smooth boundary, and suppose

$$u \in C^2\big(\Omega \times (0,T)\big) \cap C^1\big(\overline{\Omega} \times (0,T)\big)$$

satisfies

$$u_t = \Delta u \qquad (x \in \Omega, 0 < t < T),$$

with either $u = 0$ or $\partial u / \partial \nu = 0$ on the boundary $\partial\Omega$, for all $0 < t < T$. Define

$$f(t) = \int_\Omega u(x,t)^2 \, dx \qquad (0 < t < T).$$

We are asked to prove that $f(t)$ is nonincreasing. It would suffice to show that $f'(t) \le 0$ for all $0 < t < T$. But

$$f'(t) = \int_\Omega \partial_t\big(u^2\big)\, dx.$$

Following the hint, we use the identity $u(u_t - \Delta u) = \frac{1}{2}\partial_t\big(u^2\big) - \operatorname{div}(u\nabla u) + |\nabla u|^2$ (we check this by expanding the right hand side using the product rule for derivatives). Integrating this over Ω, we get, since $u_t - \Delta u = 0$,

$$0 = \underbrace{\frac{1}{2}\int_\Omega \partial_t\big(u^2\big)\, dx}_{=f'(t)} - \underbrace{\int_\Omega \operatorname{div}(u\nabla u)\, dx}_{=\int_{\partial\Omega} u\nabla u\cdot\nu\, dS = 0} + \underbrace{\int_\Omega |\nabla u|^2 \, dx}_{\ge 0},$$

where for the middle term we use the divergence theorem and the assumption that either $u = 0$ or $\partial u / \partial \nu = \nabla u \cdot \nu = 0$ on $\partial\Omega$. It now follows that $f'(t) \le 0$.

$\boxed{11}$ **a)** (Exercise 4.1.8 from McOwen.) This exercise boils down to the following: Suppose u is nonconstant, C^2 and subharmonic ($\Delta u \ge 0$) in a ball $B_r(0)$, and u is C^1 on the closed ball $\overline{B_r(0)}$. Then by the maximum principle (p. 109), we know that u attains its maximum at some point x_0 on the boundary of the ball. Moreover, we assume that x_0 is the unique maximum point, so

$$u(x) < u(x_0) \qquad \text{for all } x \in \overline{B_r(0)},\ x \ne x_0.$$

This implies (we shall need this later) that there exists $\varepsilon > 0$ such that

$$u(x) \le u(x_0) - \varepsilon \qquad \text{for all } x \in B_r(0) \text{ with } |x - x_0| = r/2. \tag{35}$$

Without loss of generality, we may assume

$$x_0 = (r, 0, \dots, 0). \tag{36}$$

We are asked to prove that

$$\frac{\partial u}{\partial \nu}(x_0) > 0, \tag{37}$$

where ν denotes the exterior unit normal of the ball.
Following the hint on p. 427, we introduce a function

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2} \qquad (x \in \mathbb{R}^n),$$

where $\alpha > 0$ is a constant. Notice that

$$v(x) = 0 \qquad \text{for all } |x| = r, \tag{38}$$

and that (recalling (36))

$$\frac{\partial v}{\partial \nu}(x_0) = v_{x_1}(r,0,\dots,0) = -2\alpha r e^{-\alpha r^2}. \tag{39}$$

We calculate $\Delta v(x) = 2\alpha\left(2\alpha|x|^2 - n\right)e^{-\alpha|x|^2}$. If $x \in B_{r/2}(x_0)$, i.e., $|x - x_0| < r/2$, then

$$|x| \geq |x_0| - |x - x_0| = r - |x - x_0| \geq r - \frac{r}{2} = \frac{r}{2},$$

and therefore $\Delta v(x) \geq 2\alpha\left(2\alpha(r/2)^2 - n\right)e^{-\alpha|x|^2} > 0$, provided $\alpha > 0$ is chosen so large that $\alpha r^2/2 > n$, i.e., $\alpha > 2n/r^2$. We summarize:

$$\Delta v(x) > 0 \qquad \text{for all } x \in B_{r/2}(x_0). \tag{40}$$

Now define $w(x) = u(x) + \eta v(x)$ for $x \in \overline{B_r(0)}$, where $\eta > 0$ is chosen so small that

$$\eta v(x) \leq \varepsilon \qquad \text{for all } |x| \leq r, \tag{41}$$

where ε is as in (35).
By (40), $\Delta w \geq 0$ in $B_r(0) \cap B_{r/2}(x_0)$, so by the weak maximum principle, w attains its maximum at a point x^* on the boundary of $B_r(0) \cap B_{r/2}(x_0)$. Thus, either (i) $|x^*| = r$, in which case $w(x^*) = u(x^*)$, by (38), or (ii) $|x^*| < r$ and$|x^* - x_0| = r/2$, in which case we have $w(x^*) \leq u(x_0)$, by (35) and (41). In either case, we conclude that $w(x) \leq u(x_0) = w(x_0)$ for all x in the closure of $B_r(0) \cap B_{r/2}(x_0)$, and hence

$$0 \leq \frac{\partial w}{\partial \nu}(x_0) = \frac{\partial u}{\partial \nu}(x_0) + \eta\frac{\partial v}{\partial \nu}(x_0).$$

But in view of (39), this implies

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\eta\frac{\partial v}{\partial \nu}(x_0) \geq \eta 2\alpha r e^{-\alpha r^2},$$

and this concludes the proof.

b) Let $M = \max_{\bar{\Omega}} u$ and define

$$\Sigma = \{x \in \Omega : u(x) = M\}.$$

We claim that either $\Sigma = \Omega$ or $\Sigma = \emptyset$. In the first case u is constant equal to M. In the second case $u < \sup_\Omega u$ in Ω. Hence the strong maximum principle follows.
Proof of claim: Assume $\Sigma \neq \emptyset$ and $\Sigma \neq \Omega$. Since u is continuous, Σ is relatively closed, and hence $\Omega \setminus \Sigma$ is open. Therefore there is a ball B and a point $x_0 \in \Sigma$ such that

$$B \subset \Omega \setminus \Sigma \qquad \text{and} \qquad \partial B \cap \Sigma = \{x_0\}.$$

Since

$$\Delta u \geq 0 \quad \text{and} \quad u < u(x_0) \quad \text{in} \quad B,$$

Hopf Lemma (part a)) implies that

$$\frac{\partial u}{\partial \nu}(x_0) < 0.$$

On the other hand, since x_0 is an interior maximum point we have $Du(x_0) = 0$. This is a contradiction so either $\Sigma = \Omega$ or $\Sigma = \emptyset$.

c) Let $w = u - v$ and note that this function belong to $C^2(\Omega) \cup C(\bar{\Omega})$ and solve

$$\begin{cases} \Delta w = 0 & \text{in} \quad \Omega \\ \frac{\partial w}{\partial \nu} + \alpha(x) w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Now assume that

$$M = \max_{\bar{\Omega}} w \geq 0,$$

(otherwise consider $-w$) and let $x_0 \in \bar{\Omega}$ be such that $w(x_0) = M$. By the strong maximum principle

$$\text{either} \quad x_0 \in \partial\Omega \quad \text{or} \quad w \equiv M.$$

The first case is not possible since the Hopf lemma implies that

$$\frac{\partial w}{\partial \nu}(x_0) + \alpha(x_0) w(x_0) > 0 + \alpha(x_0) M \geq 0,$$

contradicting the boundary condition. Hence we conclude that

$$u - v \equiv M = \text{constant} \quad \text{in} \quad \bar{\Omega}.$$

If $\alpha \not\equiv 0$, then $\alpha(x_1) \neq 0$ for some $x_1 \in \partial\Omega$. The boundary condition then implies

$$0 = \frac{\partial M}{\partial \nu} + \alpha(x_1) M = \alpha(x_1) M,$$

and hence $M = 0$.

12 In this exercise, A, B, C and R denote real $n \times n$-matrices.

We say that $A = (a_{ij})$ is *positive semi-definite* if

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq 0 \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{42}$$

Note that the left side can be written

$$\xi^T A \xi$$

if we regard ξ as a column vector, and ξ^T denotes the transpose.

a) Suppose if A is positive semi-definite. Applying (42) with ξ equal to $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$ etc., we get $a_{ii} \geq 0$ for $i = 1, \ldots, n$. Now assume λ is an eigenvalue of A, associated to an eigenvector x, so $Ax = \lambda x$. We may assume $|x| = 1$. Then $\lambda = \lambda x^T x = x^T(\lambda x) = x^T A x \geq 0$.

b) Assume A and B are symmetric and positive semi-definite. We are asked to prove that $\operatorname{tr}(AB) \geq 0$, where tr denotes the trace.

Since A is symmetric, there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of A. Then the matrix P with rows $v_1, \ldots, v_n$ diagonalizes A:

$$P^T A P = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where λ_j is the eigenvalue associated to v_j (so $A v_j = \lambda_j v_j$). Diagonalize A using an orthonormal basis of eigenvectors. Using the general fact that $\operatorname{tr}(R^t C R) = \operatorname{tr}(C)$ for all C if R is an orthogonal matrix, we get (since P is orthogonal, so in particular, $P P^T = I$)

$$\operatorname{tr}(AB) = \operatorname{tr}(P^T A B P) = \operatorname{tr}(P^T A I B P) = \operatorname{tr}([P^T A P][P^T B P]) = \sum_{j=1}^{n} \lambda_j b'_{jj},$$

where the b'_{ij}'s are the entries of the matrix $B' = P^T B P$. But B' is positive semi-definite, since B is, so by part (a), $b'_{jj} \geq 0$. By part (a) we also get $\lambda_j \geq 0$, since A is positive semi-definite. Therefore, $\operatorname{tr}(AB) \geq 0$.

13 Let $u : \Omega \to \mathbb{R}$ be C^2. We are asked to prove that if u has a local maximum at at point $x_0 \in \Omega$, then the symmetric $n \times n$-matrix $D^2 u(x_0)$ with entries $\partial_i \partial_j u(x_0)$ is negative semi-definite.

Given any $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, we define $\phi(t) = u(x_0 + t\xi)$ for t in a sufficiently small interval around 0 (so that $x_0 + t\xi$ stays inside Ω). By the chain rule (we write $\partial_i u = u_{x_i}$)

$$\phi'(t) = \nabla u(x_0 + t\xi) \cdot \xi = \sum_{i=1}^{n} \partial_i u(x_0 + t\xi)\xi_i,$$

and

$$\phi''(t) = \sum_{i,j=1}^{n} \partial_j \partial_i u(x_0 + t\xi)\xi_i \xi_j.$$

But ϕ has a local maximum at $t = 0$, hence $\phi''(0) \leq 0$, by one-variable calculus. This proves that $D^2 u(x_0)$ is negative semi-definite.

14 The purpose of this exercise is to prove the weak maximum principle (cf. (16) in Section 4.1 of McOwen) for a more general elliptic operator than the Laplace operator. So let Ω be a bounded domain in $\mathbb{R}^n$, and let

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i \partial_j + \sum_{i=1}^{n} b_i(x)\partial_i,$$

where a_{jk} and b_j are continuous functions on $\overline{\Omega}$ and the matrix $A = (a_{ij})$ is symmetric (so $a_{ij} = a_{ji}$) and pointwise positive definite, i.e.,

$$\xi^T A(x)\xi = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j > 0 \qquad \text{for all } x \in \overline{\Omega} \text{ and all } \xi \in \mathbb{R}^n \text{ with } \xi \neq 0. \tag{43}$$

(Thus, the operator L is elliptic.)

a) Assume $v \in C^2(\Omega)$ satisfies $Lv > 0$ in Ω. We are asked to prove that v cannot have a local maximum in Ω.

To get a contradiction, we assume v does have a local maximum at some point $x^* \in \Omega$. Then $\nabla v(x^*) =$, so

$$Lv(x^*) = \sum_{i,j=1}^{n} a_{ij}(x^*)\partial_i \partial_j v(x^*) = \operatorname{tr}\bigl(A(x^*)D^2 v(x^*)\bigr),$$

where $D^2 v$ is the symmetric matrix with entries $\partial_i \partial_j v$. But by the previous exercise, we know that $D^2 v(x^*)$ is negative semi-definite, hence $-D^2 v(x^*)$ is positive semi-definite. Since also $A(x^*)$ is symmetric and positive semi-definite, it follows from part (b) of exercise 13 that $\operatorname{tr}\bigl(A(x^*)[-D^2 v(x^*)]\bigr) \geq 0$, i.e., $\operatorname{tr}\bigl(A(x^*)D^2 v(x^*)\bigr) \leq 0$. Thus, we have a contradiction to $Lv(x^*) > 0$.

b) We are asked to show that if $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and $M > 0$ is sufficiently large, then

$$w(x) = \exp(-M|x - x_0|^2)$$

satisfies $Lw > 0$ in Ω. We calculate

$$\partial_i w(x) = -2M(x - x_0)_i \exp(-M|x - x_0|^2),$$

where $(x - x_0)_i$ denotes the i-th component of $x - x_0$. Further,

$$\partial_j \partial_i w(x) = \bigl(-2M\delta_{ij} + 4M^2(x - x_0)_i(x - x_0)_j\bigr)\exp(-M|x - x_0|^2),$$

where $\delta_{ij} = 0$ if $i \neq j$ and $= 1$ if $i = j$. Thus,

$$Lw(x) = 2M\left(-\operatorname{tr} A(x) + 2M(x - x_0)^T A(x)(x - x_0) - \sum_{i=1}^{n} b_i(x)(x - x_0)_i\right)\exp(-M|x - x_0|^2),$$

so we have to prove that by choosing $M > 0$ large enough, we can ensure that the expression inside the parentheses is positive:

$$2Mf(x) + g(x) > 0 \qquad \text{for all } x \in \Omega, \tag{44}$$

where

$$f(x) = (x - x_0)^T A(x)(x - x_0), \qquad g(x) = -\operatorname{tr} A(x) - \sum_{i=1}^{n} b_i(x)(x - x_0)_i.$$

To prove this, note first that since g is continuous on the compact set $\overline{\Omega}$, it is bounded, so there exists $K > 0$ such that $\left|g(x)\right| \le K$ for all $x \in \overline{\Omega}$. Next, since f is continuous on the compact set $\overline{\Omega}$, it attains its minimum m at some point $x_* \in \overline{\Omega}$. But by (43), $m > 0$. Thus, $f(x) \ge m > 0$ for all $x \in \overline{\Omega}$. We conclude that

$$2Mf(x) + g(x) \ge 2Mm - K > 0$$

provided we choose $M > K/2m$. This concludes the proof that $Lw > 0$ in Ω.

c) Suppose $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and that $Lu = 0$ in Ω. We are supposed to prove the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Following the hint, we define $v = u + \varepsilon w$, where w is as in part (b) (fix any point x_0 not in Ω) and $\varepsilon > 0$ is arbitrary. Since v is continuous on the compact set $\overline{\Omega}$, it attains its maximum at some point $x^* \in \overline{\Omega}$. By linearity of L,

$$Lv = Lu + \varepsilon Lw = \varepsilon Lw > 0 \qquad \text{in } \Omega,$$

where the last step follows from part (b). Thus, by part (a), we cannot have $x^* \in \Omega$, hence $x^* \in \partial\Omega$. We conclude that

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v.$$

But

$$\max_{\overline{\Omega}} u < \max_{\overline{\Omega}} v,$$

since $w > 0$, and

$$\max_{\partial\Omega} v \le \max_{\partial\Omega} u + \varepsilon R,$$

where $R > 0$ is chosen so large that $w(x) \le R$ for all $x \in \partial\Omega$. We conclude that

$$\max_{\overline{\Omega}} u < \max_{\partial\Omega} u + \varepsilon R$$

and letting $\varepsilon \to 0$, we obtain

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u.$$

Since the reverse inequality is obvious, we are done.