SOLUTIONS to Exam in TMA4305 Partial Differential Equations, 27.05.2008

Problem 1

a) By the method of characteristics,

$$\begin{cases} \dot{x} = 3z, & x(0) = x_0 \\ \dot{z} = 0, & z(0) = h(x_0) \end{cases} \underset{\text{integrate}}{\Longrightarrow} \begin{cases} x = 3zt + x_0 \\ z = h(x_0) \end{cases} \underset{h(x_0)=\frac{5}{3}x_0-1}{\Longrightarrow} \begin{cases} x_0 = \frac{x+3t}{5t+1} \\ z = \frac{\frac{5}{3}x-1}{5t+1}, \end{cases}$$

and the solution is

$$z = u(x,t) = \underline{\frac{\frac{5}{3}x - 1}{5t+1}}.$$

b) By the computations in a) we have the following characteristics,

$$\begin{cases} x = 3zt + x_0 & = \begin{cases} x_0, & x_0 < 0 \\ 3t + x_0, & x_0 > 0, \end{cases} \\ \\ z = h(x_0) & = \begin{cases} 0, & x_0 < 0 \\ 1, & x_0 > 0, \end{cases} \end{cases}$$

and the solution is not defined in the wedge $0 < x < 3t$. In this case a weak shock solution will be a solution of the form

$$\underline{u(x,t) = \begin{cases} 0, & x < \xi(t) \\ 1, & x > \xi(t), \end{cases}}$$

where the shock curve ξ satisfy the Rankine-Hugoniot condition

$$\dot{\xi} = \frac{G(u_r) - G(u_l)}{u_r - u_l} \qquad \text{for} \qquad G(r) = \frac{3}{2}r^2,\ u_r = 1,\ u_l = 0.$$

Note that $\partial_x G(u) = G'(u)u_x = 3uu_x$. Initially the shock is at $(0,0)$ so

$$\begin{cases} \dot{\xi}(t) = \frac{3}{2} \\ \xi(0) = 0 \end{cases} \quad \Longrightarrow \quad \underline{\xi = \frac{3}{2}t}.$$

Problem 2

a) Bilinear form $B(u,v) = \iint_\Omega [u_x v_x + 5u_y v_y - b u_x v]$.
A weak solution of (2) is a function $u \in H_0^1(\Omega)$ satisfying

$$B(u,v) = F(v) := \iint_\Omega fv \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

Note: Boundary conditions are incorporated in the space $H_0^1(\Omega)$.

b) We show existence and uniqueness using the Lax Milgram theorem. We must check that the assumptions are satisfied:

1. $X = H_0^1(\Omega)$ is a Hilbert space (ok).
2. $B : X \times X \to \mathbb{R}$ is well-defined, bounded, and coercive bilinear form.
   Well-defined and bilinear is obvious, and B is bounded since by Cauchy-Schwarz

   $$|B(u,v)| \leq \|u_x\|_2 \|v_x\|_2 + 5\|u_y\|_2\|v_y\|_2 + \|b\|_\infty \|u_x\|_2 \|v\|_2 \leq (5 + \|b\|_\infty)\|u\|_{1,2}\|v\|_{1,2}.$$

   Since

   $$\begin{aligned} B(u,u) &= \|u_x\|_2^2 + 5\|u_y\|_2^2 + \iint_\Omega b u_x u \\ &\geq \|\nabla u\|_2^2 - \|b\|_\infty \|u_x\|_2 \|u\|_2 \qquad \text{(Cauchy-Schwarz)} \\ &\geq (1 - \|b\|_\infty C_\Omega^{1/2})\|\nabla u\|_2^2 = \epsilon \|\nabla u\|_2^2 \qquad (\text{Poincare: } \|u\|_2^2 \leq C_\Omega \|\nabla u\|_2^2), \end{aligned}$$

   it follows that B is coercive when $\epsilon > 0$.
3. $F : X \to \mathbb{R}$ is well-defined, linear, and bounded.
   Well-defined and linear is obvious, and F is bounded since by Cauchy-Schwarz,

   $$|F(v)| \leq \|f\|_2 \|v\|_2 \leq \|f\|_2 \|v\|_{1,2}.$$

Hence we conclude by Lax Milgram that there is a unique $u \in H_0^1(\Omega)$ such that

$$B(u,v) = F(v) \qquad \text{for all} \qquad v \in H_0^1(\Omega),$$

and by a) this is the unique weak solution of (2).

Problem 3

a) Since the integrand is $C^1(\bar{\Omega}\times(0,\infty))$, we may interchange differentiation and integration. We then get:

$$\begin{aligned}
\frac{d}{dt}E_u(t) &= \iint_\Omega (u_t u_{tt} + c^2(u_x u_{xt} + u_y u_{yt})) \\
&= \iint_\Omega [u_t u_{tt} + c^2(\partial_x(u_x u_t) - u_{xx}u_t) + c^2(\partial_y(u_y u_t) - u_{yy}u_t)] \qquad \text{(product rule)} \\
&= \iint_\Omega u_t[u_t - c^2(u_{xx} + u_{yy})] + c^2 \iint_\Omega \operatorname{div}(\nabla u u_t) \\
&= \iint_\Omega u_t[u_t - c^2(u_{xx} + u_{yy})] + c^2 \int_{\partial\Omega} (\nabla u u_t)\cdot \nu \qquad \text{(divergence theorem)} \\
&= -\iint_\Omega u_t^2 + c^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial \nu} \le 0. \qquad \text{(equation+boundary condition)}
\end{aligned}$$

b) Assume there are two solutions u, v. Then $w = u - v$ solve

$$(*) \qquad \begin{cases} w_{tt} + w_t - c^2(w_{xx} + w_{yy}) = 0 & \text{in} \quad \Omega\times(0,\infty), \\ \frac{\partial w}{\partial \nu} = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ w = 0 \quad \text{and} \quad w_t = 0 & \text{on} \quad \bar{\Omega}\times\{0\}. \end{cases}$$

By a), $\frac{d}{dt}E_w(t) \le 0$, and by the initial conditions in (*),

$$w_t \equiv 0, w_x \equiv 0, w_y \equiv 0 \quad \text{at } t = 0 \quad \Rightarrow \quad E_w(0) = 0.$$

Hence $(0 \le) E_w(t) \le 0$, and since w is C^2,

$$E_w(t) \equiv 0 \quad \Rightarrow \quad w_t \equiv 0, w_x \equiv 0, w_y \equiv 0 \quad \Rightarrow \quad w \equiv \text{constant}.$$

Since $w(x,0) = 0$, $w \equiv 0$ and $u \equiv v$. Solutions are unique.

Problem 4

a) The Euler-Lagrange equation is given by

$$0 = D_v F(u) = \lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} \quad \text{for all} \quad v \in W_0^{1,3}(\Omega).$$

Note that $u, u+tv \in W_0^{1,3}(\Omega)$ for $|t|$ small implies that $v \in W_0^{1,3}(\Omega)$. A small calculation shows that

$$\begin{aligned}
&(u+tv)[(u_x+tv_x)^2 + (u_y+tv_y)^2] = \\
&u(u_x^2+u_y^2) + t\big[v(u_x^2+u_y^2) + 2u(u_x v_x + u_y v_y)\big] + t^2 u(v_x^2+v_y^2) + t^3 v(v_x^2+v_y^2),
\end{aligned}$$

and hence

$$F(u+tv) = F(u) + t \iint_\Omega \Big[\frac{1}{2}\big[v(u_x^2+u_y^2) + 2u(u_xv_x+u_yv_y)\big] + fv\Big] + \mathcal{O}(t^2).$$

The Euler-Lagrange equation is therefore

$$\text{(EL)}\quad \underline{0 = D_vF(u) = \iint_\Omega \Big[\frac{1}{2}v(u_x^2+u_y^2) + u(u_xv_x+u_yv_y) + fv\Big] \quad \text{for all} \quad v \in W_0^{1,3}(\Omega).}$$

b) Note that for any $u \in C^2(\Omega)$ and $v \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\iint_\Omega [uu_xv_x + uu_yv_y] &= \iint_\Omega \Big[\partial_x(uu_xv) - \partial_x(uu_x)v + \partial_y(uu_yv) - \partial_y(uu_y)v\Big] \\
&= -\iint_\Omega [\partial_x(uu_x) + \partial_y(uu_y)]v\,dx + \int_{\partial\Omega} \begin{bmatrix} uu_xv \\ uu_yv \end{bmatrix} \cdot \nu\, dS_x \\
&= -\iint_\Omega \Big[u(u_{xx}+u_{yy}) + (u_x^2+u_y^2)\Big]v + 0.
\end{aligned}$$

Here we used the divergence theorem and the fact that $uu_xv, uu_yv \in C_0^2(\Omega)$.
By this identity, (EL), and the fact that $C_0^\infty(\Omega) \subset W_0^{1,3}(\Omega)$, we get

$$\iint_\Omega \Big[\frac{1}{2}(u_x^2+u_y^2) - u(u_{xx}+u_{yy}) - (u_x^2+u_y^2) + f\Big]v = 0 \quad \text{for all} \quad v \in C_0^\infty(\Omega).$$

Since the integrand is continuous, the variational lemma then implies that

$$\underline{-u(u_{xx}+u_{yy}) - \frac{1}{2}(u_x^2+u_y^2) + f = 0 \quad \text{in} \quad \Omega.}$$

Problem 5

E.g. $r = 2$ will do since:

$$\begin{aligned}
\Delta w + |\nabla w| &= \Delta(u + \epsilon e^{rx}) + |\nabla(u+\epsilon e^{rx})| \\
&\geq \Delta(u+\epsilon e^{rx}) + |\nabla u| - |\nabla(\epsilon e^{rx})| = \Delta u + |\nabla u| + \epsilon e^{rx}(r^2 - r) \\
(**) \qquad &\geq 0 + \epsilon e^{rx}(r^2-r) > 0 \qquad \text{if} \qquad r > 1.
\end{aligned}$$

Let $\epsilon > 0$, $r = 2$, and x_0 be a maximum point of w in $\bar{\Omega}$:

$$w(x_0) \geq w(x) \qquad \text{for all} \qquad x \in \bar{\Omega}.$$

Such a point x_0 exists because w is continuous and $\bar{\Omega}$ is compact.

If $x_0 \in \Omega$ (interior maximum), then it follows that

$$\nabla w(x_0) = 0 \qquad \text{and} \qquad \sum_{i,j} \xi_i u_{x_i x_j} \xi_j \leq 0 \quad \text{for all} \quad \xi \in \mathbb{R}^2.$$

$$\Downarrow$$

$$\nabla w(x_0) = 0 \quad \text{and} \quad w_{xx}(x_0) \leq 0, \quad w_{yy}(x_0) \leq 0 \qquad (\text{take } \xi = (1,0) \text{ and then } \xi = (0,1))$$

$$\Downarrow$$

$$\Delta w(x_0) + |\nabla w(x_0)| \leq 0 + 0.$$

This contradicts (**) and therefore implies that $x_0 \in \partial\Omega$ and

$$\max_{\bar{\Omega}} w = \max_{\partial\Omega} w \qquad \text{for all} \qquad \epsilon > 0.$$

Using this identity and the definition of w leads to

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} w = \max_{\partial\Omega} w \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} e^{rx}.$$

Since Ω is bounded, the last term tend to 0 as $\epsilon \to 0$, and therefore

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u.$$

Since $\max_{\bar{\Omega}} u \geq \max_{\partial\Omega} u$, the weak maximum principle follows.