Cauchy problem and normal form

Cauchy problem for m-th order PDE:

$$(\mathrm{CP}) \quad \begin{cases} F(x, D^\alpha u, D^\beta u, \dots) = 0 & \text{in } \Omega \subset \mathbb{R}^n, \quad |\alpha|, |\beta|, \dots \leq m, \\ u = g_0, \ \frac{\partial u}{\partial \nu} = g_1, \dots, \ \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = g_{k-1} & \text{on } S \subset \Omega, \quad k \leq m. \end{cases}$$

Non-characteristic hypersurface S w.r.t. (CP):
$D^\gamma u|_S$ can be determind from (CP) for all $\gamma \in \mathbb{N}^n$.

Example: See McOwen p. 44.

New coordinates: $\tilde{x}(x) : S \to \tilde{S} \subset \{\tilde{x} : \tilde{x}_n = 0\}$ (near $x_0 \in S$)

- $u(x) = \tilde{u}(\tilde{x}(x))$, $\quad \frac{\partial u}{\partial x_i} = \sum_j \frac{\partial \tilde{u}}{\partial \tilde{x}_j} \frac{\partial \tilde{x}_j}{\partial x_i}$, $\quad \frac{\partial u}{\partial \nu}|_S = \frac{\partial \tilde{u}}{\partial \tilde{x}_n}|_{\tilde{x}_n = 0}, \dots$
- $0 = F(x, D^\alpha u, \dots) =: \tilde{F}\left(\tilde{x}(x), \tilde{D}^{\bar{\alpha}} \tilde{u}, \dots, \frac{\partial^k \tilde{u}}{\partial \tilde{x}_n^k}, \dots\right)$
- Implicit function thm: $\tilde{x}(x)$ exists if S smooth+regular ($S : \phi(x) = 0, \nabla\phi|_S \neq 0$)

OBS: S non-characteristic $\Leftrightarrow \ \frac{\partial \tilde{F}}{\partial(\partial_{\tilde{x}_n}^k u)} \neq 0$ and $\bar{\alpha}_n, \bar{\beta}_n, \dots \leq k - 1, \quad (F, g_i \in C^\infty)$

- Implicit function thm: $\tilde{F}(\dots) = 0 \Leftrightarrow \partial_{\tilde{x}_n}^k u = G(\dots)$.
- S non-char., smooth, regular (F, g_i smooth): $\quad (CP) \Leftrightarrow (NF)$

Normal form of (CP):

$$(\mathrm{NF}) \quad \begin{cases} \frac{\partial^k \tilde{u}}{\partial \tilde{x}_n^k} = G(\tilde{x}, \tilde{D}^{\bar{\alpha}} \tilde{u}, \tilde{D}^{\bar{\beta}} \tilde{u}, \dots) & \text{near } \tilde{x}(x_0), \quad \bar{\alpha}_n, \bar{\beta}_n, \dots \leq k - 1, \\ \tilde{u} = \tilde{g}_0, \ \frac{\partial \tilde{u}}{\partial \tilde{x}_n} = \tilde{g}_1, \dots, \ \frac{\partial^{k-1} \tilde{u}}{\partial \tilde{x}_n^{k-1}} = \tilde{g}_{k-1} & \text{on } \{\tilde{x} : \tilde{x}_n = 0\}, \quad k \leq m. \end{cases}$$

OBS: $\tilde{g}_i, G \in C^\infty$ in (NF) $\Rightarrow$ $\tilde{D}^\alpha \tilde{u}$ known at $\tilde{x}(x_0) \in \{\tilde{x} : \tilde{x}_n = 0\}$ for all $\alpha \in \mathbb{N}^n$ ($\{\tilde{x} : \tilde{x}_n = 0\}$ non-characteristic)

Idea: Define $\tilde{u}$ by Taylor expansion near $\tilde{x}_0 := \tilde{x}(x_0)$:

$$\tilde{u}(\tilde{x}) := \sum_{\alpha \in \mathbb{N}^n} \frac{\tilde{D}^\alpha \tilde{u}(\tilde{x}_0)}{\alpha!} (\tilde{x} - \tilde{x}_0)^\alpha.$$

Q: Does the series converge? Does $\tilde{u}$ solve (NF)?

A: Yes, if $\tilde{g}_i, G$ real analytic (C^∞+ convergent Taylor series) [cf. F. John: PDEs].

OBS: If S, F, g_i real anal. + S regular, non-char., then:

- $\tilde{x}, G, \tilde{g}_i$ real anal.
- $u(x) := \tilde{u}(\tilde{x}(x))$ soves (CP) (since $\tilde{u}$ solves (NF))

Cauchy-Kowalevski theorem

S, F, g_i real anal. + S regular, non-char. near x_0 $\Rightarrow$ exists unique real anal. solution of (CP) near x_0.

Remark: Not usefull in practice!

- There may be additional non real analytic solutions!
- (CP) not well-posed in general (no cont. dependence on data, McOwen p 47)
- Too strong assumptions!