Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem Set for Week 14

1 Let X, Y be normed spaces and $L(X,Y)$ the space of bounded linear operators from X to Y.

a) Prove that the operator norm is a norm on $L(X,Y)$.

b) Prove that if Y is a Banach space then so is $L(X,Y)$ with operator norm.

2 Let Ω be a bounded domain in $\mathbb{R}^n$.

a) Prove that $(C(\bar{\Omega}), \|\cdot\|_\infty)$ is a Banach space.

b) Prove that $(C^1(\bar{\Omega}), \|\cdot\|_{1,\infty})$ is a Banach space when

$$\|u\|_{1,\infty} = \|u\|_\infty + \||\nabla u|\|_\infty.$$

c) Prove that $C(\bar{\Omega})$ is *not* a Hilbert space with inner product

$$(f,g) = \int_\Omega f(x)g(x)dx.$$

3 Prove Youngs inequality: If $a, b > 0$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Hint: $ab = e^{\ln a + \ln b}$ use convexity of $f(x) = e^x$:

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y), \quad x, y \in \mathbb{R}^n, \quad \lambda \in (0,1).$$

4 Exercise 6.1:5 in McOwen.

5 Exercise 6.1:15 in McOwen.

6 Read Lemma 1 and 2 with proofs in McOwen chapter 6.1d.