Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2009

Problem Set for Week 16

1 (Corrected version)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\|\cdot\|_{2,2}$ denote the norm in $H^2(\Omega)$, and

$$H_0^2(\Omega) = \left\{ u \in H^2(\Omega) : \text{there is a sequence } \{\phi_i\} \subset C_0^\infty(\Omega) \text{ such that } \lim \|\phi_i - u\|_{2,2} = 0 \right\}.$$

$H_0^2(\Omega)$ is a closed subspace of $H^2(\Omega)$ and hence Hilbert space with the H^2-inner product.

a) Prove the Poincare inequality for $H_0^2(\Omega)$:

$$\|u\|_2^2 + \|\nabla u\|_2^2 \le C\|\Delta u\|_2^2 \qquad \forall v \in H_0^2(\Omega).$$

Hint: You will need the inequality (which you may take for granted!)

$$\|D^2 u\|_2 \le C\|\Delta u\|_2 \qquad \forall u \in H_0^2(\Omega).$$

Here $\|D^2 u\|_2^2 = \sum_{|\alpha|=2} \|D^\alpha u\|_2^2$. The last inequality is not elementary. A proof can be found in Stein: *Singular integrals and Differentiability Properties of Functions*.

A function $u \in H_0^2(\Omega)$ is a weak solution of the following boundary value problem for the *biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{in} \quad \partial\Omega, \end{cases} \tag{1}$$

if

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^2(\Omega).$$

b) (Evans 6.6:2) If $f \in L^2(\Omega)$, prove that exists a unique weak solution of (1).

Hints: Riesz representation theorem and the Poincare inequality from (a).

2 Exercise 6.2:4 in McOwen.

3 Exercise 6.3:3 in McOwen.

4 Exercise 6.3:7 in McOwen.