Norwegian University of Science and Technology
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EXAM IN TMA4305 Partial Differential Equations

English
Thursday, May 28, 2009
9:00 – 13:00

Aids (code C): Approved calculator
Rottman: *Matematisk formelsamling*
One sheet of A4 paper stamped by the IMF, on which you can write what you want.

Results: June 19, 2009

Give mathematical arguments for all answers, and include enough details to justify the methods used.

Problem 1 Solve the initial value problem

$$xu_x - yu_y = -u, \qquad u(x,1) = h(x),$$

where $h \in C^1(\mathbb{R})$ is a given function.

Problem 2 Consider the initial value problem

$$u_t + e^u u_x = 0, \qquad u(x,0) = \begin{cases} 1 & \text{when} \quad x < 0 \\ 2 & \text{when} \quad x > 0. \end{cases}$$

Sketch the (projected) characteristic curves.

Find the *rarefaction wave* solution of this problem for $t > 0$.

Problem 3

Let Ω be a bounded domain with smooth boundary and consider the boundary value problem

$$(1) \qquad \begin{cases} \Delta u - cu = f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \lambda u = g(x) & \text{on } \partial\Omega, \end{cases}$$

where c, λ are positive constants and $f, g \in C(\bar{\Omega})$.

Prove that the solutions of (1) belonging to $C^2(\bar{\Omega})$ are unique if $c > 0$ or $\lambda > 0$.

Hint: You may for example use Green's identity.

Problem 4 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and define the diffential operator L by

$$Lu(x,y) = (1+y^2)u_{xx}(x,y) + (1+x^2)u_{yy}(x,y) + xu_x(x,y) + yu_y(x,y).$$

a) Show that the following weak maximum principle holds,

$$u \in C^2(\Omega) \cap C(\bar{\Omega}) \quad \text{satisfies} \quad Lu \geq 0 \text{ in } \Omega \quad \Longrightarrow \quad \max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Consider $v_\epsilon(x,y) = u(x,y) + \epsilon(x^2+y^2)$.

b) Consider the boundary value problem

$$(2) \qquad \begin{cases} Lu = f(x) & \text{in } \Omega, \\ u = g(x) & \text{on } \partial\Omega. \end{cases}$$

Prove that the solutions of (2) belonging to $C^2(\Omega) \cap C(\bar{\Omega})$ are unique.

Problem 5 Let $f \in L^2(\Omega) \cap C(\Omega)$ and define

$$F(u) = \int_\Omega \Big(\frac{1}{2}(\Delta u)^2 - fu\Big)\,dx \quad \text{for every} \quad u \in H_0^2(\Omega),$$

where

$$H_0^2(\Omega) = \Big\{ u \in H^2(\Omega) : \text{there is } \{\phi_i\}_i \subset C_0^\infty(\Omega) \text{ such that } \|\phi_i - u\|_{2,2} \to 0 \ \Big\}$$

is a Hilbert space with equivalent norms $\|\phi\|_{2,2} := \|\phi\|_2^2 + \|\nabla\phi\|_2^2 + \|D^2\phi\|_2^2$ and $|\phi|_{2,2}^2 := \|\Delta\phi\|_2^2$,

(3) $$|\phi|_{2,2}^2 \leq \|\phi\|_{2,2}^2 \leq C_\Omega |\phi|_{2,2}^2 \quad \text{for all} \quad \phi \in H_0^2(\Omega).$$

a) Let $u \in H_0^2(\Omega)$ satisfy

(4) $$F(u) \leq F(v) \qquad \text{for all} \qquad v \in H_0^2(\Omega).$$

Find the Euler-Lagrange (critical point) equation for u.

b) Prove that if $u \in C^4(\bar{\Omega}) \cap H_0^2(\Omega)$ satisfies (4), then u is a classical solution of the *biharmonic equation*

$$\Delta^2 u = \Delta(\Delta u) = f \quad \text{in} \quad \Omega.$$

Hint: Since $C_0^\infty(\Omega) \subset H_0^2(\Omega)$, the inequality in (4) holds for all $v \in C_0^\infty(\Omega)$.

c) Prove that $F(u)$ is *coercive*, i.e. that there are $C_1 > 0$ and $C_2 \geq 0$ such that

$$F(u) \geq C_1 \|u\|_{2,2}^2 - C_2 \quad \text{for every} \quad u \in H_0^2(\Omega).$$

Hint: The Poincaré type inequality (3) can be useful.

d) Prove that there exists a minimizer $u \in H_0^2(\Omega)$ satisfying

$$F(u) \leq F(v) \qquad \text{for all} \qquad v \in H_0^2(\Omega).$$

Hint: The direct method of calculus of variations. Take a minimizing sequence $\{u_i\}_i \subset H_0^2(\Omega)$ satisfying $F(u_i) \to \inf_{v \in H_0^2(\Omega)} F(v) =: I$ and $|F(u_i)| \leq I + 1$ for all i.