Norwegian University of Science and Technology
Department of Mathematical Sciences

Exam in TMA4305 Partial differential Equations

Suggested solutions: May 2009

Problem 1 Characteristic equations

$$\begin{cases} \dot{x} = x, & x(0) = x_0, \\ \dot{y} = -y, & y(0) = 1, \\ \dot{z} = -z, & z(0) = u(x_0, 1) = h(x_0), \end{cases}$$

with solutions

$$x = x_0 e^t, \qquad y = e^{-t}, \qquad z = h(x_0)e^{-t}.$$

In terms of (x, y) we get $x_0 = xe^{-t}$ and $e^{-t} = y$, thus,

$$u(x, y) = z(t(x, y); x_0(x, y)) = h(xy)y.$$

Problem 2 Characteristic equations

$$\begin{cases} \dot{t} = 1, & t(0) = 0, \\ \dot{x} = e^z, & x(0) = x_0, \\ \dot{z} = 0, & z(0) = u(x_0, 0) = \begin{cases} 1, & x < 0, \\ 2, & x > 0. \end{cases} \end{cases}$$

We solve for z before x, and get

$$\begin{cases} t = t, \\ x = x_0 + te^{u(x_0,0)} = \begin{cases} x_0 + te, & x_0 < 0, \\ x_0 + te^2, & x_0 > 0, \end{cases} \\ z = u(x_0, 0) = \begin{cases} 1, & x < 0, \\ 2, & x > 0. \end{cases} \end{cases}$$

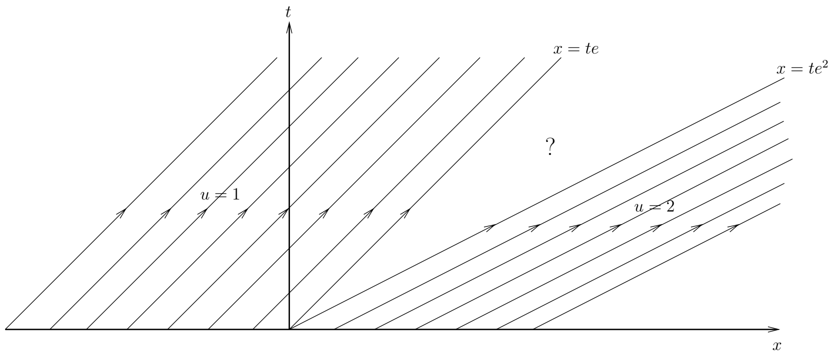

Figure 1: The projected characteristic curves of Problem 2.

See Figure 1 for a sketch of the solution.

The rarefaction fan is given by $u(x,t) = \phi(x/t)$ satisfying the PDE

$$-\frac{x}{t^2}\phi' + e^{\phi}\frac{1}{t}\phi' = 0.$$

We get $e^{\phi} = \frac{x}{t}$, that is, $\phi = \ln\frac{x}{t}$. (Note that $\phi' = 0$ implies ϕ constant and does not give a rarefaction fan.)

The total solution is

$$u(x,t) = \begin{cases} 1, & x \le te, \\ \ln\frac{x}{t}, & te \le x \le te^2, \\ 2, & x \ge te^2. \end{cases}$$

Remarks:

- Continuous, piecewise smooth solutions are weak solutions by Rankine-Hugoniot.
- This is the entropy solution since no projected characteristics collide.

Problem 3 We will need the following Green's identity (can be derived using the Divergence Theorem):

$$\int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\mathrm{d}S = \int_{\Omega}(v\Delta u + \nabla v\cdot\nabla u)\mathrm{d}x, \tag{1}$$

Assume that there are two solutions, u and v, and let $w = u - v$. Note that w is in $C^2(\Omega)$ and solves

$$\begin{aligned} \Delta w - cw &= 0, \qquad \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} + \lambda w &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

By Green's identity (1) we have

$$\int_{\partial\Omega} w \frac{\partial w}{\partial \nu} = \int_{\Omega} (w\Delta w + |\nabla w|^2),$$

and using equation (2) we get

$$\begin{gathered} -\int_{\partial\Omega} \lambda w^2 = \int_{\Omega} (cw^2 + |\nabla w|^2), \\ \int_{\Omega} (cw^2 + |\nabla w|^2) + \int_{\partial\Omega} \lambda w^2 = 0. \end{gathered}$$

Since $w \in C^2(\overline{\Omega})$, this implies that

$$\begin{cases} cw^2 + |\nabla w|^2 & = 0, \qquad \text{in } \Omega, \\ \lambda w^2 & = 0, \qquad \text{on } \partial\Omega. \end{cases} \tag{3}$$

Recall that c and λ are non-negative and $w \in C(\overline{\Omega})$. If $c + \lambda > 0$, it then follows from (3) that $w \equiv 0$ in $\overline{\Omega}$. Hence, $u \equiv v$ in $\overline{\Omega}$ and the solutions are unique.

Problem 4

a) 1. Observe that in Ω

$$Lv_\epsilon = Lu + \epsilon L(x^2 + y^2) \geq 0 + \epsilon \left(2(1 + y^2) + 2(1 + x^2) + 2x^2 + 2y^2\right) > 0. \tag{4}$$

2. Since v_ϵ is continous and $\overline{\Omega}$ compact, there is $(\overline{x}, \overline{y}) \in \overline{\Omega}$ so that $v_\epsilon(\overline{x}, \overline{y}) = \max_{\overline{\Omega}} v_\epsilon$.
3. If $(\overline{x}, \overline{y}) \in \Omega$, then at $(\overline{x}, \overline{y})$

$$v_{\epsilon\, x} = 0 = v_{\epsilon\, y}, \qquad \sum_{i=1}^{2} \xi_i v_{\epsilon\, x_i x_j} \xi_j \leq 0 \quad \text{for all } \xi \in \mathbb{R}^2.$$

In particular, take first $\xi = (1, 0)$ and then $\xi = (0, 1)$ to get

$$v_{\epsilon\, xx} \leq 0, \qquad v_{\epsilon\, yy} \leq 0 \quad \text{at } (\overline{x}, \overline{y})$$

Hence,

$$Lv_\epsilon(\overline{x},\overline{y}) \leq (1+\overline{x}^2)\cdot 0 + (1+\overline{y}^2)\cdot 0 + 0 + 0 = 0,$$

which contradicts (4). Therefore, $(\overline{x},\overline{y}) \in \partial\Omega$, and for $(x,y)\in\Omega$

$$v_\epsilon(x,y) \leq v_\epsilon(\overline{x},\overline{y}) = \max_{\partial\Omega} v_\epsilon \tag{5}$$

4. We now have

$$u(x,y) \leq v_\epsilon(x,y) \leq \max_{\partial\Omega} v_\epsilon \leq \max_{\partial\Omega}(u+\epsilon(x^2+y^2)) \leq \max_{\partial\Omega} u + 2\epsilon(\mathrm{diam}\Omega)^2.$$

Sending $\epsilon \to 0$, we get

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u.$$

b) Assume there are two solutions, $u, v \in C^2(\Omega)\cap C(\overline{\Omega})\cdot$ Let $w = u - v$ and note that $w \in C^2(\Omega)\cap C(\overline{\Omega})$ solves

$$\begin{aligned} Lw &= 0, \quad && \text{in } \Omega, \\ w &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Hence, the assumptions in *a)* are satisfied by $\pm w$, and the weak maximum principle yields

$$\begin{aligned} \max_{\overline{\Omega}} w &= \max_{\partial\Omega} w = 0, \\ \max_{\overline{\Omega}}(-w) &= \max_{\partial\Omega}(-w) = 0. \end{aligned}$$

Therefore, $|w| = 0$ in Ω and $u \equiv v$, which proves uniqueness.

Problem 5

a)

$$\begin{aligned} F(u+tv) &= \int \frac{1}{2}(\Delta(u+tv))^2 - f(u+tv) \\ &= \int \frac{1}{2}\Delta u + t\Delta u\Delta v + \frac{1}{2}t^2\Delta v^2 - fu + tfv \\ &= F(u) + t\int(\Delta u\Delta v - fv) + \frac{1}{2}t^2\int \Delta v^2. \end{aligned}$$

$$D_vF(u) = \lim_{t\to 0}\frac{F(u+tv)-F(u)}{t} = \int_\Omega(\Delta u\Delta v - fv).$$

The Euler-Lagrange equation:

$$0 = D_v F(u) = \int_\Omega (\Delta u \Delta v - fv), \qquad \text{for all } v \in H_0^2(\Omega).$$

b) By $a)$ u satisfy

$$\int_\Omega \Delta u \Delta v = \int_\Omega fv, \quad \text{for all } v \in H_0^2(\Omega)$$

Since $C_0^\infty(\Omega) \subset H_0^2(\Omega)$;

$$\int_\Omega \Delta u \Delta \phi = \int_\Omega f\phi, \quad \text{for all } \phi \in C_0^\infty(\Omega). \tag{6}$$

Integration by parts;

$$\begin{aligned}
\int_\Omega \Delta u \Delta \phi &= \sum_i \int_\Omega \Delta u \phi_{x_i x_i} \\
&\overset{\text{i.b.p.}}{=} \sum_i \left(\int_\Omega \partial_{x_i}(\Delta u \phi_{x_i}) - \int_\Omega \Delta u_{x_i} \phi_{x_i} \right) \\
&\overset{\text{div. thm.}}{=} \sum_i \left(\int_\Omega \Delta u \phi_{x_i} v_i - \int_\Omega \Delta u_{x_i} \phi_{x_i} \right) \\
&\overset{\phi \in C_0^\infty}{=} 0 - \sum_i \int_\Omega \Delta u_{x_i} \phi_{x_i} \\
&= -\sum_i \left(\int_\Omega \partial_{x_i}(\Delta u_{x_i} \phi) - \int_\Omega \Delta u_{x_i x_i} \phi \right) \\
&= 0 + \sum_i \int_\Omega \Delta u_{x_i x_i} \phi \\
&= \int_\Omega \Delta^2 u \phi.
\end{aligned}$$

By (6),

$$\int_\Omega (\Delta^2 u - v)\phi = 0, \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

and since $\Delta^2 u - v$ is continous, it follows from the variational lemma that $\Delta^2 u = v$ in Ω.

c)

$$\begin{aligned}
F(u) &= \frac{1}{2}|u|^2_{2,2} - \int fv \\
&\overset{\text{eq. (6)}}{\geq} \frac{1}{2C_\Omega}||u||^2_{2,2} - \int fv \\
&\overset{ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b}{2}}{\geq} \frac{1}{2C_\Omega}||u||^2_{2,2} - \frac{1}{2\epsilon}||v||^2_2 - \frac{\epsilon}{2}||f||^2_2 \\
&\overset{\epsilon = 2C_\Omega}{=} \frac{1}{4}||u||^2_{2,2} - C_\Omega||f||^2_2,
\end{aligned}$$

hence, F is coersive with $C_2 = \frac{1}{4}$ and $C_2 = C_\Omega||f||^2_2$.

d) The direct method:

1. Since F is coersive;

$$I := \inf_{v \in H_0^2} F(v) \geq C_1||u||^2_{2,2} - C_2 \geq -C_2 > -\infty.$$

2. Take a minimixing sequence $\{u_i\} \subset H_0^2(\Omega)$ satisfying $F(u_i) \to I$ and $|F(u_i)| \leq I+1$ for all i.
3. $||u||_{2,2} \leq M$ for all i since

$$C_1||u||^2_{2,2} \leq C_2 + F(u_i) \leq C_2 + I + 1 =: M^2 \leq \infty.$$

By weak compactness there exists a subsequence $\{u_{i_k}\} \subset \{u_i\}$, $u \in H_0^2(\Omega)$, so that $u_{i_k} \rightharpoonup u$ in $H_0^2(\Omega)$.
4. $F(u) \leq \liminf_{k\to\infty} F(u_{i_k})$: We will use that in any Hilbert space $(X, ||\cdot||)$

$$u_i \rightharpoonup u \qquad \Longrightarrow \qquad ||u|| \leq \liminf ||u_{i_k}||,$$

and that

$$u_i \rightharpoonup u \text{ in } L^2 \qquad \Longrightarrow \qquad \int fu_i \to \int fu.$$

$$\begin{aligned}
F(u) &= \frac{1}{2}|u|^2_{2,2} - \int fu \\
&\leq \liminf_{k\to\infty} f|u_{i_k}|^2_{2,2} - \lim \int fu_{i_k} \\
&= \liminf F(u_{i_k}).
\end{aligned}$$

We can now conclued that

$$I \leq F(u) \leq \liminf F(u_{i_k}) = \lim F(u_{i_k}) = I,$$

and hence $F(u) = I$.

Conclusion: There exists a $u \in H_0^2(\Omega)$ so that $F(u) = I = \inf_{v \in H_0^2} F(v)$.