Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week 10

The exercises are from McOwen's book: *Partial differential equations.*

1 *Exercise 4.1.3.* We are looking at the Laplace equation with a mixed boundary condition:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \alpha u = \beta & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where we assume α, β are constants.

Assume $\alpha > 0$; we are asked to prove uniqueness. So suppose

$$u, v \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$$

are both solutions of (1). Set $w = u - v$. Then w satisfies

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} + \alpha w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

From Green's first identity we get

$$\int_{\partial\Omega} w \frac{\partial w}{\partial \nu}\, dS = \int_\Omega \nabla w \cdot \nabla w \, dx,$$

and using the boundary condition we then get

$$-\alpha \int_{\partial\Omega} w^2 \, dS = \int_\Omega |\nabla w|^2 \, dx.$$

But the left side is ≤ 0 and the right side is ≥ 0, so both sides must equal zero. Hence $w = 0$ in Ω, and by continuity also in $\overline{\Omega}$. This proves that $u = v$.

2 *Exercise 4.1.5.* We consider

$$\Delta u - q(x) u = 0 \qquad \text{in } \Omega, \tag{3}$$

where $q(x) \geq 0$ is assumed to be bounded and continuous in Ω. Moreover, we assume that $q(x)$ is not everywhere zero, so there is a point $x_0 \in \Omega$ such that

$$q(x_0) > 0. \tag{4}$$

We are asked to prove uniqueness of solutions in the space

$$C^2(\Omega) \cap C^1\left(\overline{\Omega}\right), \tag{5}$$

subject to either (i) $u = g$ on $\partial\Omega$, or (ii) $\partial u / \partial \nu = h$ on $\partial\Omega$. (Note that uniqueness fails for (ii) if $q = 0$, so the assumption (4) is needed.)

So assume u, v are two solutions belonging to (5). Then $w = u - v$ satisfy (3) with either (i) $w = 0$ on $\partial\Omega$, or (ii) $\partial w/\partial \nu = 0$ on $\partial\Omega$. In either case,

$$\int_{\partial\Omega} w \frac{\partial w}{\partial \nu}\, dS = 0,$$

hence we get from Green's first identity:

$$0 = \int_{\Omega} |\nabla w|^2 + w\Delta w\, dx = \int_{\Omega} |\nabla w|^2 + q w^2\, dx,$$

and since the integrand on the right side is continuous and ≥ 0, it follows that $|\nabla w|^2 + q w^2 = 0$ in Ω, hence $\nabla w = 0$ in Ω, so $w = \text{const} = C$ in Ω. But also $q(x)w(x)^2 = 0$ for all $x \in \Omega$, and taking $x = x_0$ and using (4), we conclude that $w(x_0) = 0$. Thus $C = 0$, so $w = 0$ in Ω, hence in $\overline{\Omega}$, by continuity.

$\boxed{3}$ *Exercise 4.1.6.* Assume $n \geq 3$, fix $a \in \mathbb{R}^n$, and set $v(x) = |x - a|^{2-n}$ for $x \neq a$. Thus, $v(x) = r^{2-n}$, where $r = |x - a| = \sqrt{(x_1 - a_1)^2 + \cdots (x_n - a_n)^2}$. Then

$$r_{x_i} = \frac{2(x_i - a_i)}{2\sqrt{(x_1 - a_1)^2 + \cdots (x_n - a_n)^2}} = \frac{x_i - a_i}{r},$$

so the chain rule gives

$$v_{x_i} = (2-n)r^{1-n} r_{x_i} = (2-n)r^{1-n}\frac{x_i - a_i}{r} = (2-n)r^{-n}(x_i - a_i),$$

and

$$v_{x_i x_i} = (-n)(2-n)r^{-n-1}(x_i - a_i)r_{x_i} + (2-n)r^{-n} = (-n)(2-n)r^{-n-2}(x_i - a_i)^2 + (2-n)r^{-n}.$$

Thus,

$$\Delta v = \sum_{i=1}^{n} v_{x_i x_i} = (-n)(2-n)r^{-n-2}r^2 + n(2-n)r^{-n} = 0.$$

A similar calculation works for $n = 2$, with $v(x) = \log r$.

$\boxed{4}$ *Exercise 4.1.7.*

a) We assume $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and $\Delta u = 0$ in Ω, and we have to show that

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{6}$$

(So the only difference from the standard (weak) maximum principle is that we have absolute values here.) As usual, Ω is assumed bounded.

It suffices to prove

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |u|, \tag{7}$$

since the reverse inequality is obvious.

Since $|u|$ is continuous, and since $\overline{\Omega}$ is a compact set, the maximum is attained, so there exists $x_1 \in \overline{\Omega}$ such that

$$|u(x_1)| = \max_{\overline{\Omega}} |u|.$$

Without loss of generality we may assume that $u(x_1) \geq 0$ (since otherwise, we can replace u by $-u$). Thus,

$$u(x_1) = \max_{\overline{\Omega}} |u|,$$

but this implies $\max_{\overline{\Omega}} |u| = \max_{\overline{\Omega}} u$ (why?). From the standard maximum principle, we have $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, so finally we conclude that

$$\max_{\overline{\Omega}} |u| = u(x_1) = \max_{\overline{\Omega}} u = \max_{\partial\Omega} u \leq \max_{\partial\Omega} |u|,$$

which proves (7).

b) Here we are asked to show the same thing for the unbounded set

$$\Omega = \{x \in \mathbb{R}^n : |x| > 1\},$$

again assuming $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\Delta u = 0$ in Ω, and also assuming

$$\lim_{|x|\to\infty} u(x) = 0. \tag{8}$$

So we have to prove (6); again it suffices to prove (7).

Let us first note that the maxima are attained. For the maximum over $\partial\Omega = \{x : |x| = 1\}$, this is clear, since this set is compact, but for $\overline{\Omega} = \{x : |x| \geq 1\}$ it is not completely obvious, since this set is unbounded. First of all, let us assume u is not everywhere zero (otherwise there is nothing to prove!), so

$$\alpha := \sup_{\overline{\Omega}} |u| > 0.$$

By the assumption (8), there exists $R > 0$ such that

$$|x| \geq R \implies |u(x)| \leq \alpha/2. \tag{9}$$

Now, the set

$$A_R := \{x : 1 \leq |x| \leq R\}$$

is compact, hence the maximum of $|u|$ on this set is attained; thus, there exists $x_1 \in A_R$ such that $|u(x_1)| = \max_{A_R} |u|$, and in view of (9) we must then have $\max_{\overline{\Omega}_R} |u| = \alpha$.

We conclude that the sup of $|u|$ over $\overline{\Omega}$ is actually a max, and that

$$|u(x_1)| = \max_{A_R} |u| = \max_{\overline{\Omega}} |u|. \tag{10}$$

As before, we may assume $u(x_1) \geq 0$, without loss of generality. Then from (10) we conclude:

$$u(x_1) = \max_{A_R} u. \tag{11}$$

Applying the standard maximum principle on the region A_R, and recalling (9), we find that

$$\max_{A_R} u = \max_{\partial\Omega} u,$$

so we finally conclude that

$$\max_{\overline{\Omega}} |u| = u(x_1) = \max_{A_R} u = \max_{\partial\Omega} u \leq \max_{\partial\Omega} |u|,$$

which proves (7).