Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week 11

The exercises are from McOwen's book: *Partial differential equations.*

1 *Exercise 4.2.4.*

a) We are asked to prove that $G(x,y) \le 0$ for $x, y \in \Omega$ with $x \ne y$. Fix $x \in \Omega$. Recall that

$$G(x,y) = K(x-y) + \omega_x(y),$$

where $\omega_x(y)$ satisfies

$$\begin{cases} \Delta_y \omega_x = 0 & \text{in } \Omega, \\ \omega_x(y) = -K(x-y) & \text{for } y \in \partial\Omega. \end{cases}$$

Thus, $y \mapsto G(x,y)$ is harmonic for $y \in \Omega$, $y \ne x$, and

(1) $$G(x,y) = 0 \qquad \text{for } y \in \partial\Omega.$$

Since ω_x is a bounded function, and since $K(x-y) \to -\infty$ as $y \to x$, we conclude that there exists $r > 0$ such that $\overline{B_r(x)} \subset \Omega$ and

(2) $$G(x,y) < 0 \qquad \text{for } y \in \overline{B_r(x)},\ y \ne x.$$

Thus, in the set $\Omega' \equiv \Omega \setminus \overline{B_r(x)}$, the function $y \mapsto G(x,y)$ is harmonic with boundary values ≤ 0, in view of (1) and (2), hence by the weak maximum principle, $G(x,y) \le 0$ for $y \in \Omega'$ (and hence for all $y \in \Omega$, $y \ne x$).

b) By (1) and (2), $y \mapsto G(x,y)$ is not constant in Ω', so the strong maximum principle guarantees that there is no interior maximum point, i.e., $G(x,y) < 0$ for all $y \in \Omega'$ (and hence for all $y \in \Omega$, $y \ne x$).

2 *Exercise 4.2.5.* We are going to prove that

$$\int_{\partial\Omega} H(x,y)\, dS(y) = 1 \qquad \text{for all } x \in \Omega,$$

where

$$H(x,y) = \frac{\partial}{\partial \nu_y} G(x,y) \qquad \text{for } x \in \Omega,\ y \in \partial\Omega,$$

is the Poisson kernel. (This property was used in the proof of Theorem 3.) In fact, all we have to do is apply Eq. (33) on page 118 to the constant function $u \equiv 1$.

3 *Exercise 4.2.6.* Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x, y > 0\}$. We are asked to find Green's function $G(x,y;\xi,\eta)$ for Ω. So fix $P = (x,y) \in \Omega$. We have to find a function $\omega_P(\xi,\eta)$ which is harmonic in Ω and satisfies

(3) $$\omega_P(\xi,0) = -K(x-\xi,y)$$

and

$$\omega_P(0,\eta) = -K(x, y-\eta), \tag{4}$$

for all $\xi, \eta > 0$. Recall that $K(x, y) = (1/2\pi)\log|(x, y)| = (1/2\pi)\log\sqrt{x^2+y^2}$. Think of $K(x-\xi, y-\eta)$ as the potential due to a unit positive charge at P.

To satisfy (3), we first reflect P about the x-axis, i.e., we place a negative unit charge at the point $(x,-y)$. Thus, we now have charges at (x, y) and $(x,-y)$, of opposite sign. To satisfy (4), we now reflect these charges about the y-axis, and invert the signs. Mathematically, this corresponds to setting

$$\omega_P(\xi,\eta) = -K(x-\xi,-y-\eta) + K(-x-\xi,-y-\eta) - K(-x-\xi, y-\eta),$$

corresponding to negative unit charges at $(x,-y)$ and $(-x, y)$ and a positive unit charge at $(-x,-y)$; since all these points are in $\mathbb{R}^2 \setminus \overline{\Omega}$, it is clear that ω_P thus defined is harmonic in Ω, and it is easy to check directly that (3) and (4) are both satisfied. For example,

$$\omega_P(\xi,0) = -K(x-\xi,-y) + \underbrace{K(-x-\xi,-y) - K(-x-\xi, y)}_{=0} = -K(x-\xi, y),$$

since $K(x,-y) = K(x, y)$ in general.

So the answer is

$$G(x, y;\xi,\eta) = K(x-\xi, y-\eta) - K(x-\xi,-y-\eta) + K(-x-\xi,-y-\eta) - K(-x-\xi, y-\eta).$$