Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week12

The exercises are from McOwen's book: *Partial differential equations.*

1 *Exercise 4.2.7.* Assume $u \in C(\Omega)$ and satisfies the mean value property:

$$u(x) = \frac{1}{\omega_n} \int_{|y|=1} u(x+ry)\, dS(y) \qquad \text{if } \overline{B_r(x)} \subset \Omega.$$

We are supposed to prove that $u \in C^2(\Omega)$ and that $\Delta u = 0$.

Fix a ball $B_r(x)$ such that $\overline{B_r(x)} \subset \Omega$. Using Poisson's formula on this ball, with boundary values $u|_{\partial B_r(x)}$, we can find (see Theorem 4) a harmonic function $v \in C^2\big(B_r(x)\big) \cap C\big(\overline{B_r(x)}\big)$ such that $v(z) = u(z)$ for $z \in \partial B_r(x)$.

We now claim that $u = v$ in $B_r(x)$. To see this, note that $u - v$ satisfies the mean value property (since both u and v do), hence the maximum principle holds (the proof of Theorem 3 on page 109 works for any continuous function satisfying the mean value property) for $u - v$ on $B_r(x)$, so we get $u - v \leq 0$ in $B_r(x)$. Applying the maximum principle also to $v - u$ gives $v - u \leq 0$ in $B_r(x)$, and we conclude that $u - v = 0$ in $B_r(x)$, which proves the claim.

2 *Exercise 4.2.11.* We are supposed to prove Liouville's Theorem: If $u \in C^2(\mathbb{R}^n)$ is harmonic and bounded, then u is a constant.

From Eq. (46) on page 122, we have (this comes from differentiating Poisson's formula)

$$|\nabla u(x_0)| \leq \frac{n}{a} \max_{x \in \partial B_a(x_0)} |u(x)|.$$

Since u is assumed to be bounded, we get

$$|\nabla u(x_0)| \leq \frac{C}{a}$$

for all $x_0 \in \mathbb{R}^n$ and all $a > 0$, where C is independent of x_0 and a. Thus, letting $a \to \infty$, we see that $\nabla u = 0$, hence u must be a constant.