Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2009

Solutions week 13

$\boxed{1}$ (McOwen 5.1:2)

$$(1) \qquad \begin{cases} u_t &= \Delta u & \text{in } \Omega \times (0,\infty) \\ u &= h & \text{on } \partial\Omega \times (0,\infty) \\ u &= g & \text{on } \overline{\Omega} \times \{0\} \end{cases}$$

Note that $u(x,t) = v(x,t) + w(x)$ solve (1) if v and w solve

$$\begin{cases} \Delta w &= 0 & \text{in } \Omega \\ w &= h & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} v_t &= \Delta v & \text{in } \Omega \times (0,\infty) \\ v &= 0 & \text{on } \partial\Omega \times (0,\infty) \\ v &= g - w & \text{on } \overline{\Omega} \times \{0\} \end{cases}$$

If $\{\lambda_n, \varphi_n\}_{n=1}^{\infty}$ are eigenvalues and eigenfunctions for $-\Delta$ on Ω and $g - w = \sum a_n\varphi_n$, then from McOwen,

$$v(x,t) = \sum a_n e^{-\lambda_n t}\varphi_n.$$

Note that, for $t \to \infty$,

$$|v(x,t)| \le e^{-\lambda_i t} |\sum_{n=1}^{\infty} a_n e^{-(\lambda_n - \lambda_i)t}\varphi_n| \le e^{-\lambda_i t} |\sum_{n=1}^{\infty} a_n\varphi_n| \to 0$$

where λ_i is the smallest positive eigenvalue and we assumed $|\sum_{n=1}^{\infty} a_n\varphi_n|$ to be bounded. Hence

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} (v(x,t) + w(x)) = w(x)$$

$\boxed{2}$ (McOwen 5.2:1)

Theorem 1. *If g bounded continuous function and*

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t)g(y)dy$$

where

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$$

Then,

i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

ii) $u_t = \Delta u$ *in* $(\mathbb{R}^n \times (0,\infty))$

iii) $\lim_{t\to 0} u(x,t) = g(x)$

Proof.

(a) Obs: K is C^∞ for $t > 0$

$$K_t = -\frac{n}{2}\frac{4\pi}{(4\pi t)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4t}} + \frac{1}{(4\pi t)^{\frac{n}{2}}}\frac{|x-y|^2}{4t^2} e^{-\frac{|x-y|^2}{4t}}$$

$$K_{x_i x_i} = \frac{1}{(4\pi t)^{\frac{n}{2}}}[-\frac{2(x_i - y_i)}{4t} e^{-\frac{|x-y|^2}{4t}}]_{x_i}$$
$$= \frac{1}{(4\pi t)^{\frac{n}{2}}}(-\frac{2}{4t} + \frac{4(x_i - y_i)^2}{4t}) e^{-\frac{|x-y|^2}{4t}}$$

So, $K_t - \sum_{i=1}^n K_{x_i x_i} = 0$ $(t > 0)$

(b)

$$\int_{\mathbb{R}^n} K(x, y, t) dy = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dy \underset{z=\frac{y-x}{2\sqrt{t}}}{=} \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} dz = \frac{1}{\pi^{\frac{n}{2}}} \left(\int_{\mathbb{R}} e^{-s^2} ds \right)^n = 1$$

(c)

$$\int_{|x-y|>\delta} K(x, y, t) dy \underset{z=\frac{y-x}{2\sqrt{t}}}{=} \frac{1}{\pi^{\frac{n}{2}}} \int_{|z|>\frac{\delta}{2\sqrt{t}}} e^{-|z|^2} dz \le \frac{1}{\pi^{\frac{n}{2}}} \int_{|z|>\frac{\delta}{2\sqrt{t}}} e^{-\frac{1}{2}\frac{\delta^2}{4t}} e^{-\frac{1}{2}|z|} \le \frac{e^{-\frac{\delta^2}{8t}}}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}} dz \to 0$$

as $t \to 0$ uniformly in x since $\int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}} dz < \infty$.

(d)

(i)

$$D^\alpha u(x, t) = \int_{\mathbb{R}^n} (D^\alpha_{t,x} K)(x, y, t) g(y) dy$$

is continuous for all α, hence $u \in C^\infty(\mathbb{R} \times (0, \infty))$

(ii)

$$u_t - \Delta u = \int_{\mathbb{R}^n} (K_t(x, y, t) - \Delta_x K(x, y, t)) g(y) dy = 0$$

for all $t > 0$ since $K_t(x, y, t) - \Delta_x K(x, y, t) = 0$

(iii) Using (b),

$$u(x, t) - g(x) = \int_{\mathbb{R}^n} K(x, y, t)(g(y) - g(x)) dy$$
$$= (\int_{|x-y|<\delta} + \int_{|x-y|>\delta}) K(x, y, t)(g(y) - g(x)) dy$$

that implies

$$|u(x, t) - g(x)| \le \int_{|x-y|<\delta} K(x, y, t)|g(y) - g(x)| dy + 2\|g\|_\infty \int_{|x-y|>\delta} K(x, y, t) dy$$

For all $\epsilon > 0$ take $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \frac{\epsilon}{2}$ (g continuous) and $t > 0$ small such that $\int_{|x-y|>\delta} K(x, y, t) dy < \frac{\epsilon}{2} \frac{1}{2\|g\|_\infty}$ by (c). Then,

$$|u(x, t) - g(x)| < \epsilon$$

Remark: $(iii) + (i) \Rightarrow u \in C(\mathbb{R}^n \times [0, \infty))$.

□

$\boxed{3}$ (McOwen 5.2:2)

$$u(x, t) = \int_{\mathbb{R}^n} K(x, y, t) g(y) dy$$

where g bounded and continuous.

(a)

$$|u(x, t)| \le \int_{\mathbb{R}^n} K(x, y, t)|g(y)| dy \quad \text{since } K > 0$$
$$\le \|g\|_\infty \int_{\mathbb{R}^n} K(x, y, t) dy$$
$$= \quad \|g\|_\infty \quad \text{since } \int K(x, y, t) dy = 1$$

(b) Assume in addition $\int_{\mathbb{R}^n} |g(y)|dy < \infty$, then

$$\begin{aligned} |u(x,t)| &\le \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} |g(y)|dy \\ &\le \frac{\int_{\mathbb{R}^n} |g(y)|dy}{(4\pi t)^{\frac{n}{2}}} \quad \text{since } e^{-\frac{|x-y|^2}{4t}} \le 1 \end{aligned}$$

and the last term goes to zero as $t \to \infty$.

$\boxed{4}$ (McOwen 5.2:5)

Theorem 2. *Assume $u \in C(U_T \cup \Gamma_T) \cap C^{2,1}(U_T) \cap L^\infty(U_T)$ and $u_t - \Delta u \le 0$ in $U_T := \Omega \times (0,T)$ where $\Gamma_T = \Omega \times \{0\} \bigcup \partial\Omega \times (0,T)$. Then,*

$$\sup_{U_T} u = \sup_{\mathbb{R}^n} u(x,0)$$

Proof.

1) Let $\tau < T$, $\epsilon > 0$, $k > 0$:

$$w(x,t) = u(x,t) - \epsilon|x|^2 - kt$$

Obs:

$$w_t - \Delta w \le 2n\epsilon - k < 0$$

if $k > 2n\epsilon$.

2) Obs:

$$\lim_{|x|\to\infty} w(x,t) = -\infty$$

Take $R > 0$ such that

$$|x| > R \Rightarrow \epsilon R^2 > 2\|u\|_\infty + kT + 1 \Rightarrow w(x,t) < -\|u\|_\infty - 1$$

On the other hand at $(x,t) = (0,0)$

$$w(0,0) = u(0,0) \ge -\|u\|_\infty$$

Conclusion:

$$\sup_{U_\tau} w = \sup_{B_R(0)\times[0,\tau)} u = \max_{\overline{B_R(0)}\times[0,\tau]} u$$

3) Let $(x,t) \in \overline{U}_\tau$ such that

$$w(x,t) = \max_{\overline{U}_T} w$$

If $0 < t < \tau$, then $w_t = 0$ and $\Delta w \le 0$. If $t = \tau$, then $w_t \ge 0$ and $\Delta w \le 0$. Both cases are in contradiction with the observation in (1). Hence,

$$\max_{\overline{U}_\tau} w = \max_{\mathbb{R}^n} w(x,0)$$

(4) Let $(x,t) \in U_T$: Then $(x,t) \in U_\tau$ for some $\tau < T$ and

$$u(x,t) = w(x,t) + \epsilon|x|^2 + kt \le \max_{\mathbb{R}^n} w(x,0) + \epsilon|x|^2 + kT \le \sup_{\mathbb{R}^n} u(x,0) + \epsilon|x|^2 + kT,$$

where the last inequality follows since $w \le u$. Send $\epsilon \to 0$, then $k \to 0$:

$$u(x,t) \le \sup_{\mathbb{R}^n} u(x,0)$$

and hence

$$\sup_{U_T} u \le \sup_{\mathbb{R}^n} u(x,0).$$

□