Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2009

Problem set for week 14

1 We are asked to prove that the operator norm *is* a norm on the space $L(X,Y)$ of bounded linear maps from X to Y, and moreover, if Y is a Banach space, then so is $L(X,Y)$.

So let $T \in L(X,Y)$. The operator norm is defined by

$$\|T\| = \sup_{x\in X,\, x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X},$$

which is a well-defined non-negative real number, since by assumption there exists *some* constant $C > 0$ such that $\|Tx\|_Y \le C\|x\|_X$, hence $\|T\| \le C$.

Clearly,

$$\|Tx\|_Y \le \|T\|\,\|x\|_X. \tag{1}$$

The properties of a norm are easily checked. First, if $\|T\| = 0$, then (1) implies $T = 0$. Second,

$$\|cT\| = \sup_{x\in X,\, x\neq 0} \frac{\|cTx\|_Y}{\|x\|_X} = \sup_{x\in X,\, x\neq 0} \frac{|c|\,\|Tx\|_Y}{\|x\|_X} = |c| \sup_{x\in X,\, x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = |c|\,\|T\|,$$

where we used the general fact that the supremum commutes with multiplication by a non-negative constant. Third, if $S, T \in L(X,Y)$, then

$$\sup_{x\in X,\, x\neq 0} \frac{\|Sx+Tx\|_Y}{\|x\|_X} \le \sup_{x\in X,\, x\neq 0} \frac{\|Sx\|_Y + \|Tx\|_Y}{\|x\|_X} \le \sup_{x\in X,\, x\neq 0} \frac{\|Sx\|_Y}{\|x\|_X} + \sup_{x\in X,\, x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X},$$

which proves $\|S+T\| \le \|S\| + \|T\|$. This concludes the proof that $\|T\|$ is a norm.

Now assume Y is Banach, and let us prove that $L(X,Y)$ is then also Banach. So assume $\{T_j\}$ is a Cauchy sequence in $L(X,Y)$. For any $x \in X$, we have by (1) and linearity,

$$\left\|T_j x - T_k x\right\|_Y \le \left\|T_j - T_k\right\|\,\|x\|_X, \tag{2}$$

hence $\{T_j x\}$ is a Cauchy sequence in Y, hence it has a limit in Y, which we denote Tx. Doing this for every $x \in X$, we obtain a map $T : X \to Y$, and this map is linear, since each T_j is. It remains to prove that $T \in L(X,Y)$ and that $\left\|T_j - T\right\| \to 0$ as $j \to \infty$.

But being a Cauchy sequence, T_j is bounded, i.e., there exists $M > 0$ such that $\left\|T_j\right\| \le M$ for all j. Thus, $\left\|T_j x\right\|_Y \le M\|x\|_X$ for all $x \in X$, which implies that $\|Tx\|_Y \le M\|x\|_X$, hence $T \in L(X,Y)$, and $\|T\| \le M$.

Finally, given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\left\|T_j - T_k\right\| \le \varepsilon$ for all $j, k \ge N$. Then letting $k \to \infty$ in (2), we get

$$\left\|T_j x - Tx\right\| \le \varepsilon\|x\|_X \qquad \text{for all } j \ge N \text{ and all } x \in X,$$

hence $\left\|T_j - T\right\| \le \varepsilon$ for $j \ge N$.

2 **a)** Let $\overline{\Omega}$ be a bounded set, $\left(C(\overline{\Omega})\,;\quad \|f\|_\infty = \max_{x\in\overline{\Omega}} |f(x)|\right)$ is a Banach space.

Proof.

1) $\|\cdot\|_\infty$ is well-defined, indeed:

$$f \in C(\overline{\Omega}) \Rightarrow \exists\, x_0 \in \overline{\Omega} \text{ such that } \|f\|_\infty = |f(x_0)|$$

2) $\|\cdot\|_\infty$ is a norm on $C(\overline{\Omega})$. Trivial, for example,

$$\|f\|_\infty = 0 \Rightarrow |f(x)| = 0 \text{ for all } x \Rightarrow f = 0$$

3) Completeness:

$$\{f_i\} \subset C(\overline{\Omega}) \text{ Cauchy} \Rightarrow f_i(x) \subset \mathbb{R} \text{ Cauchy},$$

since

$$|f_i(x) - f_j(x)| \leq \|f_i - f_j\|_\infty.$$

This implies that there exists $y_x \in \mathbb{R}$ ($\mathbb{R}$ is complete) such that

$$f_i(x) \to y_x \quad \text{for all} \quad x \in \overline{\Omega}$$

Define the function f as

$$f(x) = y_x \quad \text{for all} \quad x \in \overline{\Omega},$$

and note that

$$\{f_i\} \text{ Cauchy} \quad \Rightarrow \quad \|f_i\|_\infty \leq M < \infty\ \forall\ i \quad \Rightarrow \quad |f(x)| \leq M\ \forall\ x \quad \Rightarrow \quad \sup_{\overline{\Omega}} |f(x)| \leq M,$$

and

$$\sup_{\overline{\Omega}} |f_j(x) - f(x)| = \sup_{\overline{\Omega}} |f_j(x) - \lim_k f_k(x)| = \lim_k \sup_{\overline{\Omega}} |f_j(x) - f_k(x)| = \lim_k \|f_j - f_k\| \to 0$$

as $j \to 0$. Hence,

(i) $f_j \to f$ uniformly in $\overline{\Omega}$ implies $f \in C(\overline{\Omega})$

(ii) $\|f_j - f\|_\infty \to 0$ as $j \to 0$

and $C(\overline{\Omega})$ is complete.

□

b) Let Ω be a bounded set, then

$$\left(C^1(\overline{\Omega})\ ; \quad \|f\|_{1,\infty} := \sup_{x \in \Omega} \left\{|f(x)| + |\nabla f(x)|\right\}\right)$$

is a Banach space.

Proof.

1) $\|\cdot\|_{1,\infty}$ well defined and norm on $C^1(\overline{\Omega})$ as in the previous exercise.

2)

$$\begin{aligned} \{f_i\} \text{ Cauchy} &\Rightarrow \{f_i(x)\}, \{\nabla f_i(x)\} \text{ Cauchy} \\ &\Rightarrow \text{ exists } y_x \in \mathbb{R},\ \overrightarrow{y}'_x \in \mathbb{R}^n \text{ such that } f_i(x), \nabla f_i(x) \to\ y_x, \overrightarrow{y}'_x \end{aligned}$$

3) Def. $f(x) = y_x$, $\overrightarrow{g}(x) = \overrightarrow{y}'_x$, $x \in \Omega$. As in the previous exercise:

$$f \in C(\overline{\Omega}),\ \|f_i - f\|_\infty \to 0$$

$$\overrightarrow{g} \in [C(\overline{\Omega})]^n,\ \|\nabla f_i - \overrightarrow{g}\|_\infty \to 0$$

4) Check: $\nabla f = \overrightarrow{g}$. For any $\epsilon > 0$,

$$\left|\frac{f(x + he_j) - f(x)}{h} - g_j(x)\right| \leq |D_{e_j} f(x) - D_{e_j} f_k(x)| + |D_{e_j} f_k(x) - g_j(x)| < \epsilon$$

since the first term on the right-hand side is less than $\frac{\epsilon}{2}$ for k large enough and the second term is less than $\frac{\epsilon}{2}$ for h small enough.

5) Conclude $f_j \to f$ in $C^1(\overline{\Omega})$ and the space is complete.

$\square$

c) Let Ω be a bounded set,

$$\left(C(\overline{\Omega})\ ; \quad < f, g >= \int_\Omega f g \right)$$

is an inner product space which is not a Hilbert space.

Proof.

1) $< \cdot, \cdot >$ well defined on $C(\overline{\Omega})$ (if Ω nice)
2) $< \cdot, \cdot >$ inner product on $C(\overline{\Omega})$. Trivial, e.g.

$$< f, f >= 0 \Rightarrow \int |f|^2 dx = 0 \Rightarrow f \equiv 0$$

2) Not complete. Counter-example in $C([-1,1])$. Define

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad \text{and} \qquad f_k(x) = \begin{cases} 0 & x < -\frac{1}{k} \\ x + \frac{1}{k} & -\frac{1}{k} \leq x < 0 \\ 1 & x \geq 0, \end{cases}$$

and observe that

$$\|f - f_k\|^2 = \int_{-1}^{1} |f - f_k|^2 = \int_{-\frac{1}{k}}^{0} |f - f_k|^2 \leq \frac{1}{k} \to \infty$$

since $|f - f_k|^2 \leq 1$. Hence $\|f_k - f\| \to 0$ but $f \not\in C(\overline{\Omega})$, so $C(\overline{\Omega})$ is not complete.

$\square$

$\boxed{3}$ (Young's inequality) Let $a, b > 0$, $p > 1$ $q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Note that the function $f(x) = e^x$ is convex. Then,

$$\begin{aligned} ab = e^{\ln a + \ln b} &= e^{\frac{1}{p} ln(a^p) + \frac{1}{q} ln(b^q)} \\ &= f(\lambda \ln a^p + (1-\lambda) \ln a^q), \quad \lambda = \frac{1}{p}, \\ &\leq \lambda f(\ln a^p) + (1-\lambda) f(\ln a^q), \quad \text{by convexity,} \\ &= \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

$\square$

$\boxed{4}$ (McOwen 6.1:5) We set $\Omega = (0,1)$.

a) Define $u : \Omega \to \mathbb{R}$ by

$$u(x) = \begin{cases} x & \text{for } 0 < x \leq 1/2, \\ 1 - x & \text{for } 1/2 \leq x < 1. \end{cases}$$

We compute the weak derivative $u'(x)$. So let $v \in C_0^\infty(\Omega)$, and calculate, using integration by parts,

$$\begin{aligned}\int_0^1 u(x)v'(x)\,dx &= \int_0^{1/2} xv'(x)\,dx + \int_{1/2}^1 (1-x)v'(x)\,dx \\ &= -\int_0^{1/2} v(x)\,dx - \int_{1/2}^1 (-1)v(x)\,dx \\ &= -\int_0^1 f(x)v(x)\,dx,\end{aligned}$$

where

$$f(x) = \begin{cases} 1 & \text{for } 0 < x \le 1/2, \\ -1 & \text{for } 1/2 \le x < 1. \end{cases}$$

This shows that f is the weak derivative of u on Ω.

b) Now set $u(x) = x^\alpha$ for $x \in \Omega = (0,1)$, where $\alpha \in \mathbb{R}$. We want to check for which α we have $u \in H^1(\Omega)$. To do this, let us use the fact that

$$H^1(\Omega) = W^{1,2}(\Omega) = \left\{f \in L^2(\Omega) : f' \in L^2(\Omega)\right\},$$

where f' denotes the distributional derivative (the weak derivative).
First, we check for which α we have $u \in L^2(\Omega)$. We have

$$\int_0^1 u(x)^2\,dx = \int_0^1 x^{2\alpha}\,dx,$$

and this integral converges if and only if $2\alpha > -1$, i.e., $\alpha > -1/2$.
Next, we check for which α we have $u' \in L^2(\Omega)$. Since $u'(x) = \alpha x^{\alpha-1}$, we see as above that this is in $L^2(\Omega)$ if and only if $\alpha - 1 > -1/2$, i.e., $\alpha > 3/2$.
So the answer is: $u \in H^1(\Omega)$ if and only if $\alpha > 3/2$.

5 (McOwen 6.1:15)
A bounded bilinear form on a Hilbert space X is a map $B : X \times X \to \mathbb{R}$ such that

(i) $B(ax + by, z) = aB(x,z) + bB(y,z)$

(ii) $B(x, ay + bz) = aB(x,y) + bB(x,z)$

(iii) $|B(x,y)| \le C\|x\|\|y\|$

for all $a, b \in \mathbb{R}$ and $x, y, z \in X$.

Thm 1. *There exists a unique bounded linear operator $A : X \to X$ such that*

$$B(x,y) = < Ax, x >, \quad < \cdot,\cdot > \text{ inner product in } X,$$

for all $x, y \in X$.

Proof.

(i) Define $F_x : X \to \mathbb{R}$ as

$$F_x(y) = B(x,y) \quad \text{for all} \quad y \in X.$$

Note that F_x is linear

$$F_x(ay + bz) = aF_x(y) + bF_x(z) \qquad \text{(by (ii))}$$

and bounded

$$|F_x(y)| = |B(x,y)| \le C\|x\|\|\|y\| \qquad \text{(by (iii))}.$$

(ii) Riesz representation theorem: There exists a unique $z_x \in X$ such that $F_x(y) =< z_x, y >$ for all $y \in X$. Moreover, $\|F_x\| = \|z_x\|$.

(iii) Define $A : X \to X$ by

$$Ax = z_x.$$

A linear:

$$\begin{aligned}
< A(ax+by), z > =< z_{ax+by}, z >= F_{ax+by}(z) &= B(ax+by, z) \\
&= aF_x(z) + bF_y(z) \quad \text{by } (i) \\
&=< az_x + bz_y, z > \quad \text{by definition of } z_x \\
(*) \qquad &=< aA_x + bA_y, z > \quad \text{by definition of } A
\end{aligned}$$

Take

$$\begin{aligned}
z = A(ax+by) - [aA_x + bA_y] \text{ in } (*) \quad &\Rightarrow \quad \|A(ax+by) - [aA_x + bA_y]\| = 0 \\
&\Rightarrow \quad A(ax+by) = aA_x + bA_y
\end{aligned}$$

A bounded:

$$\|Ax\| = \|z_x\| = \|F_x\| = \sup_{y\neq 0} \frac{|F_x(y)|}{\|y\|} = \sup_{y\neq 0} \frac{|B(x,y)|}{\|y\|} \leq C\|x\| \quad \text{by } (iii).$$

A unique: Assume $\widetilde{A}$ is such that $B(x, y) =< \widetilde{A}x, y >$ for all $x, y \in X$. Then

$$\begin{aligned}
&0 =< Ax, y > - < \widetilde{A}x, y >=< (A - \widetilde{A})x, y > \\
&\Rightarrow 0 = \|(\widetilde{A} - A)x\| \quad \text{for all } x \in X \quad (\text{take } y = (\widetilde{A} - A)x) \\
&\Rightarrow (\widetilde{A} - A)x = 0 \quad \text{for all } x \in X \\
&\Rightarrow \widetilde{A} = A
\end{aligned}$$

□