TMA4305 Partial Differential Equations Spring 2009

Solutions Week 16

$\boxed{1}$ (Corrected version)

a) The easiest way to solve this problem is to check that (do it!) from the definitions of H_0^2 and H_0^1 that

$$u \in H_0^2(\Omega) \quad \Longrightarrow \quad u, u_{x_1}, \dots, u_{x_n} \in H_0^1(\Omega),$$

and then use the Poincare inequality in $H_0^1(\Omega)$,

$$\|u\|_2^2 \le C\|\nabla u\|_2^2 \qquad \forall v \in H_0^1(\Omega).$$

The result is the following inequality:

$$\begin{aligned}
&\|u\|_2^2 + \|u_{x_1}\|_2^2 \cdots + \|u_{x_n}\|_2 \le (1+C)\|u_{x_1}\|_2^2 \cdots + (1+C)\|u_{x_n}\|_2 \\
&\le C(1+C)(\|u_{x_1x_1}\|_2^2 + \|u_{x_1x_2}\|_2^2 + \cdots + \|u_{x_1x_n}\|_2^2) + \\
&\quad \cdots + C(1+C)(\|u_{x_nx_1}\|_2^2 + \|u_{x_nx_2}\|_2^2 + \cdots + \|u_{x_nx_n}\|_2^2) \\
&= C(1+C)\|D^2u\|_2^2.
\end{aligned}$$

Now we can conclude from the inequlity $\|D^2u\|_2 \le C\|\Delta u\|_2$.

b) The idea is to prove that $(H_0^2(\Omega), (\cdot,\cdot))$ is a Hilbert space when

$$(u,v) = \int_\Omega \Delta u \Delta v,$$

and that

$$F(v) = \int_\Omega f v$$

is a bounded linear functional on $(H_0^2(\Omega), (\cdot,\cdot))$ when $f \in L^2(\Omega)$.
We can then use Riesz representation theorem to conclude that there is a unique $u \in H_2^2(\Omega)$ such that

$$(u,v) = F(v) \quad \text{for all} \quad v \in H_0^2(\Omega),$$

and hence there is a unique weak solution of (1).

1) $(\cdot,\cdot)$ is an inner product on $H_0^2(\Omega)$: Let $u, v, w \in H_0^2(\Omega)$, $a, b \in \mathbb{R}$, then
   i) $(u,u) \ge 0$
   ii) $0 = (u,u) = \|\Delta u\|_2^2 \underset{\text{exercise 1}}{\Longrightarrow} \|u\|_2 = 0 \Longrightarrow u = 0 \; a.e. \text{ in } \Omega \Longrightarrow u = 0 \text{ in } H_0^2(\Omega).$
   iii) $(au + bv, w) = a(u,w) + b(v,w)$
   iv) $(u,v) = (v,u)$
2) The induced norm $|u|^2 = (u,u)$ is equivalent to the H^2 norm $\|\cdot\|_{2,2}$, and hence $(H_0^2(\Omega), |\cdot|_{2,2})$ is complete since $(H_0^2(\Omega), \|\cdot\|_{2,2})$ is complete:
   It is obvious that
   $$|u|_{2,2} \le \|u\|_{2,2} \quad \text{for} \quad u \in H_0^2(\Omega).$$
   To prove the opposite inequality, we need the Poincare type inequalities from Exercise 1 and the Hint:
   $$\|u\|_2^2 + \|\nabla u\|_2^2 + \|D^2u\|_2^2 \le C\|\Delta u\|_2^2 \quad \text{for} \quad u \in H_0^2(\Omega).$$
   By these inequalities it follows that
   $$\|u\|_{2,2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2 + \|D^2u\|_2^2 \le C\|\Delta u\|_2^2 = C|u|_{2,2}^2,$$
   and hence the norms are equivalent.

3) F is a bounded linear functional: This is obvious e.g.

$$|F(v)| \le \|f\|_2 \|v\|_2 \le K \|f\|_2 |v|_{2,2}.$$

$\boxed{2}$ *Exercise 6.2.4.* For a bounded domain $\Omega \subset \mathbb{R}^n$, we set

$$\lambda_1 = \inf_{u \in C_0^\infty(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega u(x)^2\, dx}.$$

a) We prove $\lambda_1 > 0$. In fact, by the Poincaré inequality,

$$\int_\Omega u(x)^2\, dx \le C \int_\Omega |\nabla u(x)|^2\, dx,$$

for all $u \in C_0^\infty(\Omega)$, where $C > 0$ only depends on Ω. So if $u \neq 0$ (so that $\int_\Omega u(x)^2\, dx > 0$), then we have

$$\frac{1}{C} \le \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega u(x)^2\, dx},$$

which implies $\lambda_1 \ge 1/C > 0$.

b) We consider the Dirichlet problem

$$\begin{cases} \Delta u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ is given and $c < \lambda_1$ is a constant.

We are asked to prove the existence of a weak solution $u \in H_0^1(\Omega)$. By definition, this must satisfy

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v - cuv\, dx}_{=B(u,v)} = -\underbrace{\int_\Omega f v\, dx}_{=F_f(v)} \qquad \text{for all } v \in C_0^\infty.$$

So the existence follows from the Riesz representation theorem, provided we can show that $B(u,v)$ is an inner product on $H_0^1(\Omega)$, whose associated norm $\sqrt{B(u,u)}$ is equivalent to the standard norms $\|\cdot\|_{1,2}$ and $|\cdot|_{1,2}$ on $H_0^1(\Omega)$ (recall that the latter two are equivalent by Poincaré's inequality).

Clearly, $B(u,v)$ is symmetric, and it is linear in both u and v. It remains to obtain upper and lower bounds on $B(u,u)$. By density, it suffices to do this for $u \in C_0^\infty(\Omega)$, $u \neq 0$. Since

$$-\int_\Omega u(x)^2\, dx \ge -\frac{1}{\lambda_1} \int_\Omega |\nabla u(x)|^2\, dx,$$

we have

$$B(u,u) \ge \varepsilon \int_\Omega |\nabla u|^2\, dx = \varepsilon\, |u|_{1,2}^2, \qquad \text{where} \qquad \varepsilon = \left(1 - \frac{c}{\lambda_1}\right) > 0. \tag{2}$$

On the other hand, it is obvious that

$$B(u,u) \le (1 + |c|)\, \|u\|_{1,2}^2. \tag{3}$$

From (2) we conclude that B satisfies the positivity required of an inner product, and from (2) and (3) together we conclude that the norm $\sqrt{B(u,u)}$ is equivalent to the standard norms $\|\cdot\|_{1,2}$ and $|\cdot|_{1,2}$ on $H_0^1(\Omega)$.

$\boxed{3}$ *Exercise 6.3.3.* Suppose H is a Hilbert space, and that $x_n \to x$ weakly in H. Moreover, suppose that $\|x_n\| \to \|x\|$. We are going to prove that this implies strong convergence $x_n \to x$.

Indeed, we have

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2.$$

By assumption, the first term on the right hand side converges to $\|x\|^2$. By weak convergence, the second term converges to $-2\langle x, x\rangle = -2\|x\|^2$. We conclude that $\|x_n - x\|^2 \to 0$, hence $\|x_n - x\| \to 0$, which means precisely that $x_n \to x$ strongly.

$\boxed{4}$ *Exercise 6.3.7.* If suffices to consider $H = l^2(\mathbb{N})$, since every separable Hilbert space is isometrically isomorphic to this space.

So assume $\{x_j\} \subset l^2(\mathbb{N})$ is bounded. I.e., writing $x_j = \{\alpha_n^j\}_{n=1}^{\infty} \in l^2(\mathbb{N})$, there exists a constant $C > 0$ such that

$$\|x_j\|^2 = \sum_{n=1}^{\infty} (\alpha_n^j)^2 \le C^2, \tag{4}$$

for all j. In particular, this implies that

$$\left|\alpha_n^j\right| \le C \qquad \text{for all } j \text{ and } n.$$

Therefore, starting with $n = 1$, we can find a sequence

$$j_1^1 < j_2^1 < \cdots < j_k^1 < \cdots \tag{5}$$

in $\mathbb{N}$ such that $\alpha_1^{j_k^1}$ converges to a number α_1 as $k \to \infty$.

Next, take $n = 2$. Since $\alpha_2^{j_k^1} \le C$ for all k, we can then find a sequence

$$j_1^2 < j_2^2 < \cdots < j_k^2 < \cdots \tag{6}$$

which is a *subsequence* of (5), and such that $\alpha_2^{j_k^2}$ converges to a number α_2 as $k \to \infty$.

Continuing in this way, we obtain an infinite matrix

$$\begin{matrix} j_1^1 & j_2^1 & j_3^1 & \cdots \\ j_1^2 & j_2^2 & j_3^2 & \cdots \\ j_1^3 & j_2^3 & j_3^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

with the properties: (i) each row is a subsequence of the previous row, and (ii) $\lim_{k\to\infty} \alpha_n^{j_k^n} = \alpha_n$.

Now we apply the diagonal argument, defining $\beta_n^k = \alpha_n^{j_k^k}$. Then by the properties (i) and (ii),

$$\lim_{k\to\infty} \beta_n^k = \alpha_n \tag{7}$$

for all n.

Define

$$y_k = \{\beta_n^k\}_{n=1}^{\infty}.$$

Thus, $y_k = x_{j_k^k}$, so $\{y_k\}$ is a subsequence of $\{x_j\}$.

From (4) we get, letting $j \to \infty$ along the subsequence j_k^k,

$$\sum_{n=1}^{N} \alpha_n^2 \le C^2,$$

for all N, hence

$$\sum_{n=1}^{\infty} \alpha_n^2 \le C^2.$$

Defining

$$x = \{\alpha_n\},$$

we then have $x \in l^2(\mathbb{N})$.

To finish up, we show that $y_k \to x$ weakly in $l^2(\mathbb{N})$. To this end, let $z = \{\gamma_n\} \in l^2(\mathbb{N})$. We have to prove that $\langle y_k, z \rangle \to \langle x, z \rangle$ as $k \to \infty$, i.e.,

$$\sum_{n=1}^{\infty} \left[\beta_n^k - \alpha_n\right] \gamma_n \to 0 \qquad \text{as } k \to \infty. \tag{8}$$

So let $\varepsilon > 0$. For arbitrary N, we have by the Cauchy-Schwarz inequality and (4),

$$\sum_{n=N+1}^{\infty} \beta_n^k \gamma_n \le \sqrt{\sum_{n=N+1}^{\infty} (\beta_n^k)^2} \sqrt{\sum_{n=N+1}^{\infty} \gamma_n^2} \le C \sqrt{\sum_{n=N+1}^{\infty} \gamma_n^2}.$$

Now choose N so large that

$$\sqrt{\sum_{n=N+1}^{\infty} \gamma_n^2} \le \frac{\varepsilon}{C}.$$

Then we get

$$\sum_{n=N+1}^{\infty} \beta_n^k \gamma_n \le \varepsilon \qquad \text{for all } k. \tag{9}$$

Similarly we obtain

$$\sum_{n=N+1}^{\infty} \alpha_n \gamma_n \le \varepsilon. \tag{10}$$

By (7) we have

$$\lim_{k\to\infty} \left(\sum_{n=1}^{N} \left[\beta_n^k - \alpha_n\right] \gamma_n \right) = 0. \tag{11}$$

Combining (9), (10) and (11), we conclude that (8) holds.