Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2009

Solutions for week 17

1

a) Consider $g \in H^1(\Omega)$. The set

$$\mathscr{A} = \{u \in H^1(\Omega) : u - g \in H_0^1(\Omega)\}$$

is weakly closed in $H^1(\Omega)$.

Proof. $\mathscr{A}$ weakly closed in $H^1(\Omega)$ means that

$$\mathscr{A} \ni u_k \rightharpoonup u \quad \text{in} \quad H^1(\Omega) \qquad \Rightarrow \qquad u \in \mathscr{A}.$$

Note that

$$\begin{aligned} &\mathscr{A} \ni u_k \rightharpoonup u \quad \text{in} \quad H^1(\Omega) \\ &\Downarrow \\ &H_0^1(\Omega) \ni (u_k - g) \rightharpoonup (u - g) \quad \text{in} \quad H_0^1(\Omega) \\ &\Downarrow \\ &H_0^1(\Omega) \ni (u_k - g), \quad \|u_k - g\|_{1,2} \le M \quad \text{for all} \quad k \\ &\Downarrow \\ &\text{There exists } \{u_{k_j} - g\} \subset \{u_k - g\}, w \in H_0^1(\Omega) \quad \text{such that} \quad u_{k_j} - g \rightharpoonup w \end{aligned}$$

But, since $(u_{k_j} - g) \rightharpoonup (u - g)$ and the weak limit is unique,

$$u - g = w \in H_0^1(\Omega) \Rightarrow u \in \mathscr{A}$$

□

b) Define

$$F(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + f u)$$

Consider a function $g \in H^1(\Omega)$ and the set

$$\mathscr{A} = \{u \in H^1(\Omega) : u - g \in H_0^1(\Omega)\}$$

Then,

$$F(v) \ge C_1 \|v\|_{1,2}^2 - C_2$$

for all $v \in \mathscr{A}$ and a $C_1 > 0$.

Proof. Obs 1: For $\epsilon > 0$ and $a, b \in \mathbb{R}$,

$$(\overline{a} - \overline{b}) \ge 0 \quad \Leftrightarrow \quad 2\overline{a}\overline{b} \le \overline{a}^2 + \overline{b}^2 \quad \Rightarrow \quad 2ab \le \frac{a^2}{\epsilon} + \epsilon b^2 \quad \text{when} \quad \overline{a} = \frac{a}{\sqrt{\epsilon}}, \overline{b} = \epsilon b^2.$$

Obs 2: $\|u\| \le \|u - g\| + \|g\|$ and $\|g\| \le \|u - g\| + \|u\|$. So,

$$\|u - g\|^2 \ge (\|u\| - \|g\|)^2 = \|u\|^2 - 2\|u\|\|g\| + \|g\|^2.$$

Hence by Obs 1 with $\epsilon = \frac{1}{2}$,

$$\|u - g\|^2 \ge \frac{1}{2}\|u\|^2 - \|g\|^2$$

Therefore,

$$\begin{aligned}
\int |\nabla u|^2 &\geq \frac{1}{2}\int |\nabla(u-g)|^2 - \int |\nabla g|^2 && \text{(Obs 2 with } u+g \text{ instead of } u\text{)} \\
&\geq \frac{1}{2}\frac{1}{1+C_\Omega}\|u-g\|_{1,2}^2 - \|g\|_{1,2} && \text{(Poincare: } \|\phi\|_2^2 \leq C_\Omega \|\nabla\phi\|_2^2,\ \phi\in H_0^1(\Omega)\text{)} \\
&\geq \frac{1}{4}\frac{1}{1+C_\Omega}\|u\|_{1,2}^2 - \Big(\frac{1}{2}\frac{1}{1+C_\Omega}+1\Big)\|g\|_{1,2}^2 && \text{(Obs 2 again).}
\end{aligned}$$

Let

$$K = \frac{1}{4}\frac{1}{1+C_\Omega} \qquad \text{and} \qquad \overline{K} = \Big(\frac{1}{2}\frac{1}{1+C_\Omega}+1\Big),$$

and note that from Obs 1 with $\epsilon = \frac{K}{2}$ we get

$$\int |fu| \leq \frac{\|f\|_2^2}{2\epsilon} + \frac{\epsilon}{2}\|u\|_2^2 \leq \frac{\|f\|_2^2}{2\epsilon} + \frac{\epsilon}{2}\|u\|_{1,2}^2 = \frac{K}{4}\|u\|_{1,2} + \frac{\|f\|_2^2}{K}.$$

Therefore, using the previous two relationships on $\int |\nabla u|^2$ and $\int |fu|$ and the fact that $u \in \mathscr{A}$,

$$\begin{aligned}
F(u) &\geq \frac{1}{2}\int |\nabla u|^2 - \int |fu| \\
&\geq \frac{1}{2}K\|u\|_{1,2}^2 - \overline{K}\|g\|_{1,2}^2 - \frac{1}{4}K\|u\|_{1,2}^2 - \frac{1}{K}\|f\|_2^2 \\
&\geq \frac{1}{4}K\|u\|_{1,2}^2 - (\overline{K}\|g\|_{1,2}^2 + \frac{1}{K}\|f\|_2^2)
\end{aligned}$$

□

2 *Exercise 7.1.6.* Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, $q : \Omega \to \mathbb{R}$ is a bounded function, with upper bound $\eta \geq 0$, i.e., $q(x) \leq \eta$ for all $x \in \Omega$, and $f \in L^2(\Omega)$. We consider the Dirichlet problem

$$\begin{cases} \Delta u + qu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We are asked to prove that there exists a unique weak solution.

By the usual integration by parts argument, the natural definition of a weak solution for this problem is the following:

$$u \in X := H_0^1(\Omega)$$

is a weak solution of (1) if

$$\int_\Omega -\nabla u \cdot \nabla v + quv\, dx = \int_\Omega f v\, dx \qquad \text{for all } v \in C_0^\infty(\Omega).$$

Equivalently, in our standard notation,

$$-(u,v)_1 + \langle qu, v\rangle = \langle f, v\rangle. \tag{2}$$

Note also that by density of $C_0^\infty(\Omega)$ in X, saying that (2) holds for all $v \in C_0^\infty(\Omega)$ is equivalent to saying that it holds for all $v \in X$.

We now want to find a functional $F : X \to \mathbb{R}$ such that the Euler-Lagrange equation $D_v F(u) = 0$ is equivalent to (2). Based on our experience with the standard Poisson equation, this is not too difficult. We try

$$F(u) = \frac{1}{2}(u,u)_1 - \frac{1}{2}\langle qu, u\rangle + \langle f, u\rangle.$$

Then

$$F(u+v) - F(u) = (u,v)_1 + \frac{1}{2}(v,v)_1 + \langle f, v\rangle - \langle qu, v\rangle - \frac{1}{2}\langle qv, v\rangle \tag{3}$$

for all $u, v \in X$, hence

$$D_v F(u) = \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} = (u,v)_1 + \langle f, v \rangle - \langle qu, v \rangle, \tag{4}$$

so that $D_v F(u) = 0$ is indeed equivalent to (2).

Existence. It suffices to find a minimizer for F, i.e., a $u \in X$ such that $F(u) = I$, where $I = \inf_X F$. We first prove that F is bounded below, so that I is a well-defined real number. Using the upper bound $q(x) \le \eta$, and the inequalities of Cauchy-Schwarz and Poincaré, we have

$$\langle qu, u \rangle \le \eta \|u\|_2^2 \le C^2 \eta \|\nabla u\|_2^2 = C^2 \eta (u,u)_1, \tag{5}$$

where $C = C(\Omega) > 0$. Similarly, and using also exercise 1 above,

$$|\langle f, u \rangle| \le \|f\|_2 \|u\|_2 \le C \|f\|_2 \|\nabla u\|_2 \le \frac{C^2}{4\varepsilon} \|f\|_2^2 + \varepsilon \|\nabla u\|_2^2. \tag{6}$$

By (5) and (14), we have for every $\varepsilon > 0$,

$$F(u) \ge \left(\frac{1}{2} - \frac{1}{2} C^2 \eta - \varepsilon\right) \|\nabla u\|_2^2 - \frac{C^2}{4\varepsilon} \|f\|_2^2, \tag{7}$$

which proves that F is bounded below, provided η satisfies

$$\eta < \frac{1}{C^2}. \tag{8}$$

(Then we can choose $\varepsilon > 0$ so small that $(1/2)C^2\eta + \varepsilon < 1/2$, hence the first term on the right hand side of (7) is non-negative.)

Now let $\{u_j\} \subset X$ be a minimizing sequence for F, i.e., $F(u_j) \to I$. We can also assume $F(u_j) \le I + 1$ for all j. We show $\{u_j\}$ is bounded:

$$\begin{aligned} \|\nabla u_j\|_2^2 &= 2F(u_j) + \langle qu_j, u_j \rangle - 2\langle f, u_j \rangle && \text{by definition of } F \\ &\le 2(I+1) + C^2\eta \|\nabla u_j\|_2^2 + \frac{C^2}{2\varepsilon} \|f\|_2^2 + 2\varepsilon \|\nabla u_j\|_2^2, && \text{by (5) and (14)} \end{aligned}$$

whence

$$\left(1 - C^2\eta - 2\varepsilon\right) \|\nabla u_j\|_2^2 \le 2(I+1) + \frac{C^2}{4\varepsilon} \|f\|_2^2.$$

So assuming (8) holds, and choosing $\varepsilon > 0$ so small that $C^2\eta + 2\varepsilon < 1$, we conclude that $\{u_j\}$ is bounded in X (recall that $|u|_{1,2} = \|\nabla u\|_2$ is one of the equivalent norms that we use on X).

By the theorem about weak compactness in a Hilbert space, we can therefore assume that

$$u_j \to u \qquad \text{weakly in } X. \tag{9}$$

Moreover, by the compactness of the inclusion $H_0^1(\Omega) \subset L^2(\Omega)$, we can assume that

$$u_j \to u' \qquad \text{in } L^2(\Omega). \tag{10}$$

We must have

$$u = u', \tag{11}$$

by uniqueness of weak limits in $L^2(\Omega)$. To finish we must show $F(u) \le I$; then it follows that $F(u) = I$, i.e., u is a minimizer, hence a critical point, hence a weak solution of (1).

To this end, we need the following construction. Let us define, for all $u, v \in X$,

$$B(u,v) = (u,v)_1 - \langle qu, v \rangle.$$

We claim this is an inner product on X, whose associated norm is equivalent to the standard norms on X. Clearly, B is symmetric, and linear in both u and v. By (5),

$$B(u,u) \ge \left(1 - C^2\eta\right) \|\nabla u\|_2^2. \tag{12}$$

Let M be a bound for $|q|$, so $|q(x)| \le M$ for all $x \in \Omega$. Then using again Cauchy-Schwarz and Poincaré as in the proof of (5), we have

$$\left|\langle qu, u\rangle\right| \le M\|u\|_2^2 \le C^2 M\|\nabla u\|_2^2,$$

whence

$$B(u,u) \le \left(1 + MC^2\right) \|\nabla u\|_2^2. \tag{13}$$

From (12), (13) and (8), we conclude that B is positive definite, and the associated norm

$$\|u\| = \sqrt{B(u,u)}$$

is equivalent to the standard norms on X.

With this knowledge, we can finish the proof that u is a minimizer. We write

$$\begin{aligned}
F(u) &= \frac{1}{2}B(u,u) + \langle f, u\rangle && \text{by definition of } F \\
&\le \liminf_{j\to\infty}\left(\frac{1}{2}B(u_j,u_j)\right) + \langle f, u\rangle && \text{by (9) and exercises 5 and 2 above} \\
&= \liminf_{j\to\infty}\left(\frac{1}{2}B(u_j,u_j)\right) + \liminf_{j\to\infty}\langle f, u_j\rangle && \text{by (10) and (11), and exercise 4(ii)} \\
&\le \liminf_{j\to\infty} F(u_j) && \text{by exercise 3} \\
&= \lim_{j\to\infty} F(u_j) && \text{by exercise 4(ii)} \\
&= I.
\end{aligned}$$

This concludes the proof of weak existence.

Uniqueness. By (3), (5) and (12), and assuming (8) holds,

$$F(u+v) - F(u) - D_v F(u) = B(v,v) \ge \left(1 - C^2\eta\right)\|\nabla v\|_2^2 > 0 \tag{14}$$

for all $u, v \in X$, $v \ne 0$. Thus, we have strict convexity, which implies uniqueness.

3 (McOwen 7.1:8 b)
Let $\Omega \subset \mathbb{R}^2$ be bounded. Consider

$$F(u) = \int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy$$

where $u \in H^1(\Omega)$.

$$\mathscr{A} = \{u \in H^1(\Omega) : u - g \in H_0^1(\Omega)\} = \{u = g + v : v \in H_0^1(\Omega)\}$$

The function u has a critical point of F on $\mathscr{A}$ if

$$0 = D_v F(u) = \lim_{t\to 0} \frac{F(u+tv) - F(u)}{t}$$

for all $v \in H^1(\Omega)$ such that $u + tv \in \mathscr{A}$ for t small.

Obs: $u + tv \in \mathscr{A}$ for t small implies $u \in H_0^1(\Omega)$.

Let

$$f(p,q) = \sqrt{1+p^2+q^2}$$

and note that $f_p = \frac{p}{f(p,q)}$, $f_q = \frac{q}{f(p,q)}$, and f, f_p, f_q continuous. Formally:

$$\begin{aligned}
D_v F(u) = \frac{d}{dt} F(u+tv)\Big|_{t=0} &= \int_\Omega \frac{\partial}{\partial t} f(u_x + tv_x, u_y + tv_y)\Big|_{t=0} \\
&= \int_\Omega \Big(\frac{(u_x + tv_x) v_x}{f(u_x+tv_x, u_y+tv_y)} + \frac{(u_y+tv_y)v_y}{f(u_x+tv_x, u_y+tv_y)} \Big)\Big|_{t=0} \\
&= \int_\Omega \frac{u_x v_x + u_y v_y}{f(u_x, u_y)} = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}}
\end{aligned}$$

These computations are correct if e.g. $u, v \in C^\infty(\overline{\Omega})$ and then they hold by density for $u, v \in H^1(\Omega)$. Hence: $u \in H^1(\Omega)$ critical point of F on $\mathscr{A}$ if the Euler-Lagrange equation holds:

$$0 = D_v(F) = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

Assume $u \in C^2(\Omega)$, integration by parts in the previous expression gives

$$0 = -\int_\Omega \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) v \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

The variational lemma then implies

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = 0$$

Obs:

$$\begin{aligned}
\partial_{x_i}\Big(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\Big) &= \frac{u_{x_i x_i}}{\sqrt{1+|\nabla u|^2}} + u_{x_i} \partial_{x_i} \frac{1}{\sqrt{1+|\nabla u|^2}} \\
&= \frac{u_{x_i x_i}}{\sqrt{1+|\nabla u|^2}} - u_{x_i} \frac{1}{2} \frac{1}{(1+|\nabla u|^2)^{\frac{3}{2}}} \sum 2 u_{x_j} u_{x_j x_i} \\
&= \frac{(1+|\nabla u|^2) u_{x_i x_i} - \sum u_{x_i} u_{x_j} u_{x_j x_i}}{(1+|\nabla u|^2)^{\frac{3}{2}}}
\end{aligned}$$

Hence

$$0 = \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \frac{(1+|\nabla u|^2)\Delta u - \sum\sum u_{x_i} u_{x_j} u_{x_j x_i}}{(1+|\nabla u|^2)^{\frac{3}{2}}}$$

So, since we are in $\mathbb{R}^2$,

$$0 = (1 + u_x^2 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2 + u_y^2) u_{yy}$$

In other words, the abouve Euler Langrance equation is a weak formulation of a minimal surface equation!