Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week 5

The exercises are from McOwen's book: *Partial differential equations.*

$\boxed{1}$ *Exercise 1.2.2.* We are going to solve Burgers' equation (we use variables x, t instead of x, y)

$$u_t + uu_x = 0,$$

with initial condition

$$u(x,0) = \begin{cases} u_0 & \text{if } x \le 0, \\ u_0(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{if } x \ge 1, \end{cases}$$

where $u_0 > 0$ is a constant.

Recall that if u_l and u_r are smooth solutions of Burgers' equation, defined, respectively, to the left and right of a curve $x = \xi(t)$, then they can be patched together to make a weak solution if and only if the Rankine-Hugoniot (R-H) condition is satisfied:

$$\xi'(t)(u_r - u_l) = \frac{1}{2}u_r^2 - \frac{1}{2}u_l^2, \tag{1}$$

where u_l and u_r are evaluated at $(x = \xi(t), t)$. Note that if there is no discontinuity across $x = \xi(t)$, i.e., if $u_l = u_r$ along this curve, then R-H is trivially satisfied. If on the other hand $u_l \ne u_r$ (so there is a jump discontinuity), then R-H reduces to

$$\xi'(t) = \frac{u_l + u_r}{2}.$$

Recall also that the characteristics are straight lines

$$x = x_0 + ut, \tag{2}$$

where u is constant along each characteristic.

In the present case, we have characteristics $x = x_0 + u_0 t$ starting from points $x_0 \le 0$, and $x = x_0$ starting from points $x_0 \ge 1$. If $0 \le x_0 \le 1$, then $x = x_0 + u_0(1-x_0)t$, which gives $x_0 = (x - u_0 t)/(1 - u_0 t)$, hence the value along the characteristic starting from x_0 is $u = u_0(1-x_0) = u_0(1-x)/(1-u_0 t)$. Thus,

$$u(x,t) = \begin{cases} u_0 & \text{if } x \le u_0 t, \\ \frac{u_0(1-x)}{1-u_0 t} & \text{if } u_0 t < x < 1, \\ 0 & \text{if } x \ge 1. \end{cases} \tag{3}$$

Note that (1) is trivially satisfied along the curves $x = u_0 t$ and $x = 1$, since the solution is continuous across those curves (but not differentiable, so it is only a weak solution). However, at time $t = 1/u_0$ the two curves $x = u_0 t$ and $x = 1$ collide, so we have to start over, with initial condition coming from (3):

$$u(x, 1/u_0) = \begin{cases} u_0 & \text{if } x < 1, \\ 0 & \text{if } x > 1. \end{cases} \tag{4}$$

We get a shock $x = \xi(t)$ for $t \geq 1/u_0$, with $\xi(1/u_0) = 1$, $u = u_0$ to the left of the shock and $u = 0$ to the right. R-H then gives $\xi'(t) = u_0/2$, hence $\xi(t) = u_0 t/2 + 1/2$. We conclude that for $t \geq 1/u_0$,

$$u(x,t) = \begin{cases} u_0 & \text{if } x < 1/2 + u_0 t/2, \\ 0 & \text{if } x > 1/2 + u_0 t/2. \end{cases} \tag{5}$$

So (3) for $0 \leq t < 1/u_0$, and (5) for $t \geq 1/u_0$, together describe the solution for all $t \geq 0$.

$\boxed{2}$ *Exercise 1.2.7.* The equation is simply $\rho_t + c\rho_x = 0$, which we recognize as the transport equation.

- **a)** $\frac{d}{dt}\left[\rho(x_0 + ct, t)\right] = c\rho_x + \rho_t = 0$.
- **b)** $\rho(x,t)$ is supposed to be the density of cars on the highway at time t. One way to make this precise is to choose a reference length $L > 0$ (much bigger than the length of a typical car) and define $\rho(x,t)$ to equal the number of cars in the interval $[x-L, x+L]$ at time t (and if a car is only partially inside the interval, we still count it as one car). Then if there is a single car on the road, which we may assume occupies the interval $0 \leq x \leq a$ at time $t = 0$ (where $a \ll L$), then $\rho(x,0)$ will be zero except for $-L \leq x \leq a + L$, where $\rho(x,0) = 1$. By part (a), this density profile will shift to the right with speed c as time increases, giving $\rho(x,t)$.
- **c)** A car alone on the road moves with speed c.

$\boxed{3}$ *Exercise 1.2.8.* Here we look at

$$\rho_t + [G(\rho)]_x = 0 \qquad \text{where} \qquad G(\rho) = c\rho\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{6}$$

Here $c > 0$ and $\rho_{\max} > 0$ are constants.

To interpret this for traffic flow, rewrite it as a continuity equation $\rho_t + (\rho v)_x = 0$, where $v = c(1 - \rho/\rho_{\max})$. Thus v is interpreted as the speed of the cars (since then ρv becomes the flux density for ρ). Note that if $\rho \ll \rho_{\max}$, then $v \approx c$, which we interpret to mean that c is the free speed of the highway, in view of the previous exercise. The cars stop moving when ρ reaches the value $\rho_{\max}$. Thus, $\rho_{\max}$ represents the maximum density of cars (bumper to bumper traffic).

Now we take initial data

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_{\max} & \text{if } x < 0, \\ \rho_{\max} & \text{if } x > 0. \end{cases}$$

Interpretation: medium traffic to the left of $x = 0$, and bumper to bumper to the right of $x = 0$, at time $t = 0$.

Note that every constant is a solution of (6). Thus, we expect to have $\rho = \frac{1}{2}\rho_{\max}$ to the left of a curve $x = \xi(t)$ and $\rho = \rho_{\max}$ to the right of this curve. This will be a weak solution of (6) if and only if the Rankine-Hugoniot condition is satisfied, which in this case becomes:

$$\xi'(t)\left(\rho_{\max} - \frac{1}{2}\rho_{\max}\right) = G(\rho_{\max}) - G(\frac{1}{2}\rho_{\max}) = 0 - \frac{1}{4}c\rho_{\max}.$$

This simplifies to $\xi'(t) = -c/2$, hence $\xi(t) = -ct/2$. We conclude that the solution is

$$\rho(x,t) = \begin{cases} \frac{1}{2}\rho_{\max} & \text{if } x < -ct/2, \\ \rho_{\max} & \text{if } x > -ct/2. \end{cases}$$

Interpretation: the back of the stopped traffic propagates with half the free speed of the highway, when the incoming traffic is at medium density.

$\boxed{4}$ *Exercise 1.2.9.* We consider the same equation as in the previous exercise, but now with initial condition

$$\rho(x,0) = \begin{cases} \rho_{\max} & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Interpretation: bumper to bumper to the left of $x = 0$, no traffic to the right of $x = 0$, at time $t = 0$. This corresponds to traffic stopped at a red light, turning green at time $t = 0$.

We should now get a rarefaction solution (the cars gradually speed up out of the green light) instead of a shock solution. To find this solution, let us first observe that (6) says that ρ is constant along characteristics. The characteristic curves $x = x(t)$ are determined by

$$\frac{dx}{dt} = G'(\rho) = c\left(1 - \frac{2\rho}{\rho_{\max}}\right), \qquad x(0) = x_0,$$

where ρ is constant. Thus,

$$x = x_0 + c\left(1 - \frac{2\rho}{\rho_{\max}}\right)t. \tag{7}$$

Referring to the initial condition, we first plug $\rho = \rho_{\max}$ into (7), obtaining characteristics $x = x_0 - ct$, for $x_0 < 0$. Next, $\rho = 0$ gives characteristics $x = x_0 + ct$ for $x_0 > 0$. That leaves open the wedge $-ct < x < ct$. The characteristics inside this wedge must start at $x_0 = 0$, so we get from (7):

$$x = c\left(1 - \frac{2\rho}{\rho_{\max}}\right)t \implies \frac{x}{ct} = 1 - \frac{2\rho}{\rho_{\max}} \implies \rho = \frac{\rho_{\max}}{2}\left(1 - \frac{x}{ct}\right).$$

We conclude that

$$\rho(x,t) = \begin{cases} \rho_{\max} & \text{if } x < -ct, \\ \frac{\rho_{\max}}{2}\left(1 - \frac{x}{ct}\right) & \text{if } -ct < x < ct, \\ 0 & \text{if } x > ct. \end{cases}$$