Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week 6

The exercises are from McOwen's book: *Partial differential equations.*

1 *Exercise 2.2.1(b).* The equation is

$$x^2 u_{xx} - y^2 u_{yy} = 0$$

Then $a = x^2$, $b = 0$ and $c = -y^2$, so $b^2 - 4ac = 4x^2y^2 > 0$ if $x, y \neq 0$, so then the equation is hyperbolic.

We transform the equation to canonical coordinates. To find the characteristic curves, in the form of graphs $y = f(x)$, we solve (see p. 50 of McOwen)

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm 2xy}{2x^2} = \pm \frac{y}{x}.$$

This separates to

$$\frac{dy}{y} = \pm \frac{dx}{x}.$$

Integration gives $\ln|y| = \pm \ln|x| + C = \ln|x|^{\pm 1} + C$, hence $y = Ax$ or $y = A/x$, where A is an arbitrary constant. We can therefore choose canonical coordinates

$$\mu = \frac{x}{y}, \qquad \eta = xy. \tag{1}$$

The chain rule gives

$$\begin{aligned}
u_x &= u_\mu \mu_x + u_\eta \eta_x = \frac{1}{y} u_\mu + y u_\eta, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = \left(-\frac{x}{y^2}\right) u_\mu + x u_\eta, \\
u_{xx} &= \frac{1}{y^2} u_{\mu\mu} + 2u_{\mu\eta} + y^2 u_{\eta\eta}, \\
u_{yy} &= \frac{2x}{y^3} u_\mu + \frac{x^2}{y^4} u_{\mu\mu} - \frac{2x^2}{y^2} u_{\mu\eta} + x^2 u_{\eta\eta}.
\end{aligned}$$

This gives $x^2 u_{xx} - y^2 u_{yy} = 4x^2 u_{\mu\eta} - \frac{2x}{y} u_\mu$, so our transformed equation is

$$u_{\mu\eta} = \frac{1}{2xy} u_\mu = \frac{u_\mu}{2\eta}.$$

Integrating in μ gives

$$u_\eta = \frac{u}{2\eta} + f(\eta), \tag{2}$$

where f is arbitrary. This is a linear first order ODE in η, which we solve using the integrating factor (let us assume $\eta > 0$ for the moment)

$$\exp\left(-\int \frac{d\eta}{2\eta}\right) = \eta^{-1/2},$$

where we chose the constant $C = 0$ in the indefinite integral. Thus, we obtain, using (2),

$$(\eta^{-1/2}u)_\eta = -\frac{u}{\eta^{3/2}} + \eta^{-1/2}u_\eta = \eta^{-1/2}f(\eta),$$

and integrating this in η, we get

$$\eta^{-1/2}u = \int \eta^{-1/2}f(\eta)\,d\eta + G(\mu),$$

for an arbitrary $G(\mu)$. Writing $F(\eta) = \eta^{1/2}\left(\int \eta^{-1/2}f(\eta)\,d\eta\right)$, we conclude:

$$u = F(\eta) + \eta^{1/2}G(\mu).$$

Substuting back the original variables x, y gives the general solution

$$\underline{\underline{u(x,y) = F(xy) + (xy)^{1/2}G(x/y)}},$$

for arbitrary functions F and G.