Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week 7

The exercises are from McOwen's book: *Partial differential equations.*

1 *Exercise 2.3.8.* We define, for each $n \in \mathbb{N}$, a function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \dfrac{n}{2} & \text{for} \quad |x| < \dfrac{1}{n}, \\ 0 & \text{for} \quad |x| \geq \dfrac{1}{n}. \end{cases}$$

We are asked to show that $f_n \to \delta$ in $\mathscr{D}'(\mathbb{R})$ as $n \to \infty$. In other words, we have to show that for every $\phi \in C_0^\infty(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\phi(x)\,dx = \langle \delta, \phi \rangle = \phi(0).$$

But

$$\int_{\mathbb{R}} f_n(x)\phi(x)\,dx = \frac{n}{2} \int_{-1/n}^{1/n} \phi(x)\,dx,$$

and by the mean value theorem for integrals,[1] there exists $x_n \in [-1/n, 1/n]$ such that

$$\int_{-1/n}^{1/n} \phi(x)\,dx = \phi(x_n)\frac{2}{n}.$$

Putting this information together, we conclude that

$$\int_{\mathbb{R}} f_n(x)\phi(x)\,dx = \phi(x_n) \longrightarrow \phi(0) \qquad \text{as } n \to \infty,$$

and the proof is complete.

2 *Exercise 2.3.10.* Let $a \in \mathbb{R}$, $a \neq 0$.

a) We are asked to find a fundamental solution for $L = d/dx - a$ on $\mathbb{R}$, i.e., to solve

$$\frac{dF}{dx} - a = \delta.$$

We proceed in the usual way to solve this linear first-order ODE: The integrating factor is e^{-ax}, and multiplying by this we rewrite the ODE as

$$\frac{d}{dx}\left(e^{-ax}F\right) = e^{-ax}\delta.$$

But $e^{-ax}\delta = \delta$ (why?), so we simply have to solve

$$\frac{d}{dx}\left(e^{-ax}F\right) = \delta.$$

[1] This states that if $g(x)$ is a continuous function on $[a, b]$, then there exists a point $c \in [a, b]$ such that

$$\int_a^b g(x)\,dx = g(c)\,(b-a).$$

Since $H' = \delta$, where H is the Heaviside function, one solution of the above ODE is clearly $e^{-ax}F = H$, i.e.,

$$F(x) = e^{ax}H(x)$$

should be a fundamental solution.
One way to check our answer is by using the fact that, by definition,

$$LF = \delta \iff \langle F, L'\phi \rangle = \phi(0) \text{ for all } \phi \in C_0^\infty(\mathbb{R}).$$

But $L' = -d/dx - a$, so defining F as above we have

$$\begin{aligned}\langle F, L'\phi \rangle &= \int_{\mathbb{R}} e^{ax}H(x)[-\phi'(x) - a]\,dx = \int_0^\infty e^{ax}[-\phi'(x) - a]\,dx \\ &= -\int_0^\infty \frac{d}{dx}\left(e^{ax}\phi(x)\right)\,dx = -\left[e^{ax}\phi(x)\right]_{x=0}^{x=\infty} = \phi(0),\end{aligned}$$

since ϕ has compact support.

b) Now $L = d^2/dx^2 - a^2$ on $\mathbb{R}$, and we are asked to show that $LF = \delta$, where

$$F(x) = \begin{cases} \dfrac{1}{a}\sinh ax & \text{for } \quad x > 0, \\ 0 & \text{for } \quad x < 0. \end{cases}$$

We know that $LF = \delta$ is equivalent to

(1) $$\langle F, L'\phi \rangle = \phi(0) \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}),$$

but in the present case, $L' = L$, so the left side is equal to

(2) $$\begin{aligned}\int_{\mathbb{R}} F(x)\left(\phi''(x) - a^2\phi(x)\right)\,dx = &= \int_0^\infty \frac{1}{a}\sinh ax\left(\phi''(x) - a^2\right)\,dx \\ &= \int_0^\infty \frac{1}{a}(\sinh ax)\phi''(x)\,dx - \int_0^\infty a(\sinh ax)\phi(x)\,dx \\ &= I + J.\end{aligned}$$

Integrating by parts, we find

$$I = -\int_0^\infty \frac{1}{a}\frac{d}{dx}(\sinh ax)\phi'(x)\,dx + \left[\frac{1}{a}(\sinh ax)\phi'(x)\right]_{x=0}^{x=\infty} = -\int_0^\infty (\cosh ax)\phi'(x)\,dx,$$

where the boundary terms vanish because $\sinh 0 = 0$ and $\phi'(x) = 0$ for all sufficiently large x. Integrating by parts once more, we get

$$I = \int_0^\infty \frac{d}{dx}(\cosh ax)\phi(x)\,dx - \left[(\cosh ax)\phi(x)\right]_{x=0}^{x=\infty} = \int_0^\infty a(\sinh ax)\phi(x)\,dx + \phi(0).$$

since $\cosh 0 = 1$ and $\phi(x) = 0$ for all sufficiently large x. But the integral on the right hand side is nothing else than $-J$ (cf. (2)), so we conclude that (1) holds, and we are done.

3 *Exercise 2.3.11.c.* Assume $f \in C_0(\mathbb{R})$, and define $u = F * f$, where $F(x) = |x|/2$, as in Example 2 on page 69. Thus,

$$u(x) = \frac{1}{2}\int_{-\infty}^\infty |x - y|\,f(y)\,dy \qquad \text{for } x \in \mathbb{R}.$$

We are asked to prove that $u \in C^2(\mathbb{R})$ (so then u is a classical solution of $u'' = f$).
But splitting the integral into $\int_{-\infty}^x (\cdots)\,dx + \int_x^\infty (\cdots)\,dx$, we get

$$u(x) = \frac{1}{2}\int_{-\infty}^x (x - y)f(y)\,dy - \frac{1}{2}\int_x^\infty (x - y)f(y)\,dy,$$

and since $f \in C_0(\mathbb{R})$, we can differentiate these integrals to get[2]

$$u'(x) = \frac{1}{2}\int_{-\infty}^{x} f(y)\,dy - \frac{1}{2}\int_{x}^{\infty} f(y)\,dy.$$

(Note that the integrals can in fact be written as integrals over bounded intervals, since f by assumption has compact support; thus, there is an $R > 0$ such that $f(x) = 0$ for all x satisfying $|x| \geq R$.) Differentiating once more, we get

$$u''(x) = \frac{1}{2}f(x) + \frac{1}{2}f(x) = f(x).$$

Thus, $u \in C^2(\mathbb{R})$ and $u'' = f$.

[2]Recall that if

$$F(x) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy,$$

where $\alpha(x)$ and $\beta(x)$ are differentiable, and $f(x,y)$ and $f_x(x,y)$ are continuous, then

$$F'(x) = \beta'(x) f\big(x,\beta(x)\big) - \alpha'(x) f\big(x,\alpha(x)\big) + \int_{\alpha(x)}^{\beta(x)} f_x(x,y)\,dy.$$