Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week 8

The exercises are from McOwen's book: *Partial differential equations.*

1 *Exercise 3.3.1.* Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and let ν denote the outward pointing unit normal on $\partial\Omega$. We consider solutions of the wave equation

$$u_{tt} = c^2 \Delta u, \qquad u = u(x,t), \qquad x \in \Omega,\ t > 0, \tag{1}$$

with either a *Dirichlet boundary condition*:

$$u(x,t) = 0 \quad \text{for all } x \in \partial\Omega,\ t \geq 0, \tag{2}$$

or a *Neumann boundary condition*:

$$\underbrace{\frac{\partial u}{\partial \nu}}_{=\nabla u \cdot \nu}(x,t) = 0 \quad \text{for all } x \in \partial\Omega,\ t \geq 0. \tag{3}$$

Here we assume

$$u \in C^2\big(\Omega \times (0,\infty)\big) \cap C^1\big(\overline{\Omega} \times [0,\infty)\big).$$

(Remark: This assumption can be weakened to $u \in C^2\big(\Omega\times(0,\infty)\big) \cap H^2(\Omega \times (0,\infty)) \cap C^1\big(\overline{\Omega}\times[0,\infty)\big)$.)

The object is to show that the energy

$$\mathscr{E}_\Omega(t) = \frac{1}{2}\int_\Omega u_t^2 + c^2 |\nabla u|^2 \, dx$$

is conserved.

By the usual integration by parts argument we have

$$\begin{aligned}
\mathscr{E}'_\Omega(t) &= \int_\Omega u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t \, dx \\
&= \int_\Omega \underbrace{u_t(u_{tt} - c^2\Delta u)}_{=0,\text{ by (1)}} \, dx + \int_{\partial\Omega} u_t (\nabla u \cdot \nu)\, dS \\
&= \int_{\partial\Omega} u_t \frac{\partial u}{\partial \nu} \, dS,
\end{aligned}$$

and the last integral vanishes under either of the assumptions (2) or (3); for the latter it is obvious, while if (2) holds, then we just have to note that $u_t = 0$ on $\partial\Omega$ in that case. Thus, $\mathscr{E}'_\Omega(t) = 0$, proving that $\mathscr{E}_\Omega(t)$ is constant.

2 *Exercise 3.3.2.* Here we are asked to prove uniqueness of solutions the boundary/initial value problem (note that in the book the initial data have been forgotten)

$$\begin{cases} u_{tt} - c^2 \Delta u = f(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = \gamma(x,t), & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \Omega, \end{cases} \tag{4}$$

where f, γ, g, h are given functions, and u is assumed to belong to the space

$$u \in C^2\big(\Omega \times (0,\infty)\big) \cap C^1\big(\overline{\Omega} \times [0,\infty)\big).$$

So now assume u, v are both in this space, and solve (4). Then $w = u - v$ also solves (4), but with $f = 0$, $\gamma = 0$, $g = 0$ and $h = 0$, so by the previous exercise, the energy in Ω must be zero for all $t \geq 0$:

$$\mathscr{E}_\Omega(t) = \mathscr{E}_\Omega(0) = 0.$$

But $\mathscr{E}_\Omega(t) = \frac{1}{2}\int_\Omega w_t^2 + c^2 |\nabla w|^2 \, dx$, so it follows that $w_t = 0$ and $\nabla w = 0$ in $\Omega \times [0,\infty)$, hence $w =$ const, and this constant must be zero, since $w = 0$ at time $t = 0$. Thus $w = 0$, i.e., $u = v$.

The same argument applies if we replace the Dirichlet boundary condition with a Neumann condition.

3 *Exercise 3.4.2.* We are asked to find dispersive solutions

$$u = e^{i(kx - \omega t)}$$

of various PDEs. In other words, we need to determine in each case a dispersion relation and solve this to find $\omega = \omega(k)$. Note that

$$u = U(kx - \omega t),$$

where $U(s) = e^{is}$, hence $U' = iU$, $U'' = -U$, $U''' = -iU$ and $U^{(4)} = U$.

a) The flexible beam equation $u_{tt} + \gamma^2 u_{xxxx} = 0$. Plugging the above u into this equation, we get

$$-\omega^2 + \gamma^2 k^4 = 0 \implies \omega = \pm\gamma k^2.$$

b) The linearized KdV equation $u_t + c u_x + u_{xxx} = 0$. With u as above we get

$$-i\omega + ick - ik^3 = 0 \implies \omega = ck - k^3 = k(c - k^2).$$

c) The Boussinesq equation $u_{tt} - c^2 u_{xx} = \gamma^2 u_{ttxx}$. Now we get

$$-\omega^2 + c^2 k^2 = \omega^2 k^2 \implies \omega^2 = \frac{c^2 k^2}{1 + k^2} \implies \omega = \pm \frac{ck}{\sqrt{1 + k^2}}$$

d) The Schrödinger equation $u_t = i u_{xx}$. Then

$$-i\omega = -ik^2 \implies \omega = k^2.$$

4 *Exercise 3.4.3.* We proceed as in the previous exercise, but now for the heat equation $u_t = u_{xx}$. Then

$$-i\omega = -k^2 \implies \omega = -ik^2.$$

Hence

$$u = e^{i(kx + ik^2 t)} = e^{i(kx + ik^2 t)} = e^{-k^2 t} e^{ikx},$$

so the solution decays exponentially as $t \to \infty$. This is therefore called a diffusive wave.