Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2009

Solutions for Problem Set Week 9

The exercises are from McOwen's book: *Partial differential equations.*

$\boxed{1}$ *Exercise 3.2.2.* We are given the Cauchy problem

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz}, \\ u(x,y,z,0) = x^2 + y^2, \quad u_t(x,y,z,0) = 0. \end{cases}$$

By Kirchhoff's formula, the solution is given by (here we denote by (ξ,η,ζ) a point on the unit sphere S^2, and dS is the surface area element on S^2)

$$\begin{aligned} u(x,y,z,t) &= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{S^2}\left[(x+t\xi)^2+(y+t\eta)^2\right]dS\right) \\ &= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{S^2}\left[x^2+2xt\xi+t^2\xi^2+y^2+2yt\eta+t^2\eta^2\right]dS\right) \\ &= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\left[4\pi(x^2+y^2)+2xt\int_{S^2}\xi\,dS+2yt\int_{S^2}\eta\,dS+t^2\int_{S^2}\xi^2+t^2\int_{S^2}\eta^2\,dS\right]\right) \end{aligned}$$

But

$$\int_{S^2}\xi\,dS = 0$$

as we can see by explicit calculation using spherical coordinates, or simply by symmetry (split S^2 into the hemispheres $\xi \ge 0$ and $\xi \le 0$; then the two integrals cancel out). Similarly, $\int_{S^2}\eta\,dS = 0$. Further, by the rotational symmetry of the sphere we have

$$\int_{S^2}\xi^2\,dS = \int_{S^2}\eta^2\,dS = \int_{S^2}\zeta^2\,dS \implies \int_{S^2}\xi^2\,dS = \frac{1}{3}\int_{S^2}(\xi^2+\eta^2+\zeta^2)\,dS = \frac{1}{3}\int_{S^2}1\,dS = \frac{4\pi}{3}.$$

We conclude that

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\left[4\pi(x^2+y^2)+t^2\frac{8\pi}{3}\right]\right) = \underline{\underline{x^2+y^2+2t^2}}.$$

(Remember: we can check our answer by plugging it into the equation; the data are certainly correct.)

For part (b) we are asked to calculate u using the 2d formula. We can do this since the data are independent of z. Denoting by (ξ,η) a point in the unit disk $D = \{(\xi,\eta) : \xi^2+\eta^2 < 1\}$ in the plane, we then have

$$\begin{aligned} u(x,y,z,t) &= \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_D \frac{(x+t\xi)^2+(y+t\eta)^2}{\sqrt{1-\xi^2-\eta^2}}\,d\xi d\eta\right) \\ &= \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_D \frac{x^2+y^2+2xt\xi+2yt\eta+t^2(\xi^2+\eta^2)}{\sqrt{1-\xi^2-\eta^2}}\,d\xi d\eta\right) \\ &= \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\left[(x^2+y^2)\int_D\frac{d\xi d\eta}{\sqrt{1-\xi^2-\eta^2}}+2xt\int_D\frac{\xi\,d\xi d\eta}{\sqrt{1-\xi^2-\eta^2}}\right.\right. \\ &\qquad\left.\left.+2yt\int_D\frac{\eta\,d\xi d\eta}{\sqrt{1-\xi^2-\eta^2}}+t^2\int_D\frac{(\xi^2+\eta^2)d\xi d\eta}{\sqrt{1-\xi^2-\eta^2}}\right]\right). \end{aligned}$$

But by symmetry,

$$\int_D \frac{\xi\, d\xi\, d\eta}{\sqrt{1-\xi^2-\eta^2}} = \int_D \frac{\eta\, d\xi\, d\eta}{\sqrt{1-\xi^2-\eta^2}} = 0.$$

Furthermore, switching to polar coordinates (r,θ) in the plane, we have

$$\int_D \frac{d\xi\, d\eta}{\sqrt{1-\xi^2-\eta^2}} \int_0^1 \int_0^{2\pi} \frac{r\, dr\, d\theta}{\sqrt{1-r^2}} = 2\pi \int_0^1 \frac{r\, dr}{\sqrt{1-r^2}} = \pi \int_0^1 \frac{ds}{\sqrt{s}} = 2\pi,$$

and

$$\int_D \frac{(\xi^2+\eta^2)\, d\xi\, d\eta}{\sqrt{1-\xi^2-\eta^2}} = \int_0^1 \int_0^{2\pi} \frac{r^2 (r\, dr\, d\theta)}{\sqrt{1-r^2}} = 2\pi \int_0^1 \frac{r^3\, dr}{\sqrt{1-r^2}} = \pi \int_0^1 \frac{(1-s)\, ds}{\sqrt{s}} = \frac{4\pi}{3}.$$

We conclude that

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\left[2\pi(x^2+y^2) + \frac{4\pi}{3} t^2\right]\right) = \underline{\underline{x^2+y^2+2t^2}}.$$

Moral: The 3d formula was easier to use (less calculations).

$\boxed{2}$ *Exercise 3.2.3.* We are asked to write the solution of the nonhomogeneous 3d wave equation

$$u_{tt} - c^2 \Delta u = f(x,t)$$

with zero initial data at time $t = 0$, using Duhamel's principle. As usual, we introduce the solution operator $S(t)$ by defining $S(t)h$ to be the solution $v(\cdot,t)$ of the homogeneous IVP

$$v_{tt} - c^2 \Delta u = 0, \qquad v(x,0) = 0, \qquad v_t(x,0) = h(x).$$

So by Kirchhoff's formula (here $x \in \mathbb{R}^3$)

$$\big(S(t)h\big)(x) = \frac{t}{4\pi} \int_{|y|=1} h(x+cty)\, dS(y).$$

Therefore, Duhamel's principle takes the form

$$\begin{aligned} u(x,t) &= \int_0^t \big(S(t-s) f(\cdot,s)\big)(x)\, ds \\ &= \int_0^t \frac{t-s}{4\pi} \left(\int_{|y|=1} f\big(x + c(t-s)y, s\big)\, dS(y) \right) ds. \end{aligned}$$

This defines u as a C^2 function, provided that $f(x,t)$ is C^2 in the x-variable and C^0 in the t-variable (then we can differentiate the integral by the usual rules).

Notice that as s ranges from 0 to t and y ranges over the sphere S^2, the point

$$\gamma(s,y) = \big(x + c(t-s)y, s\big) \in \mathbb{R}^4$$

parametrizes a cone-shaped surface Γ (the "backwards light cone") with its vertex at the point (x,t). Thus, Γ is the domain of dependence for the point (x,t): the value of the solution u at the point (x,t) depends only on the values of f on this cone Γ.

$\boxed{3}$ *Exercise 3.2.5.* We are asked to find a solution formula for the IVP for the 2d Klein-Gordon equation:

$$\begin{cases} v_{tt} - c^2 \Delta v + m^2 v = 0, & [v = v(x,y,t)], \\ v(x,y,0) = g(x,y), & v_t(x,y,0) = h(x,y). \end{cases}$$

Following the hint in the back of the book, we define

$$u(x,y,z,t) = \cos\left(\frac{m}{c} z\right) v(x,y,t).$$

A direct calculation shows that u satisfies the wave equation in 3d, so it is given by Kirchhoff's formula. Let us assume $g = 0$ for simplicity. Then, letting (ξ, η, ζ) denote a point on the unit sphere $S^2 \subset \mathbb{R}^3$, we have

$$u(x, y, z, t) = \frac{t}{4\pi} \int_{S^2} \cos\left(\frac{m}{c}(z + ct\zeta)\right) h(x + ct\xi, y + ct\eta)\, dS(\xi, \eta, \zeta).$$

Setting $z = 0$ we get

$$v(x, y, t) = u(x, y, 0, t) = \frac{t}{4\pi} \int_{S^2} \cos(mt\zeta) h(x + ct\xi, y + ct\eta)\, dS(\xi, \eta, \zeta).$$

As in the derivation of the solution formula for the wave equation in 2d we now parametrize the hemispheres $\zeta \geq 0$ and $\zeta \leq 0$ of S^2 as graphs

$$\zeta = \pm\sqrt{1 - \xi^2 - \eta^2}$$

over the unit disk $D = \{(\xi, \eta) : \xi^2 + \eta^2 \leq 1\}$; then the integral transforms to (recall that the cosine function is even, so there is no difference between the integrals over the two hemispheres!)

$$v(x, y, t) = \frac{t}{2\pi} \int_D \frac{\cos\left(mt\sqrt{1 - \xi^2 - \eta^2}\right) h(x + ct\xi, y + ct\eta)}{\sqrt{1 - \xi^2 - \eta^2}}\, d\xi\, d\eta.$$

If we now drop the assumption that $g = 0$, we get the general formula:

$$\begin{aligned} v(x, y, t) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi} \int_D \frac{\cos\left(mt\sqrt{1 - \xi^2 - \eta^2}\right) g(x + ct\xi, y + ct\eta)}{\sqrt{1 - \xi^2 - \eta^2}}\, d\xi\, d\eta\right) \\ + \frac{t}{2\pi} \int_D \frac{\cos\left(mt\sqrt{1 - \xi^2 - \eta^2}\right) h(x + ct\xi, y + ct\eta)}{\sqrt{1 - \xi^2 - \eta^2}}\, d\xi\, d\eta. \end{aligned}$$