Problem 1 Find the global solution of

$$u_t + u^3 u_x = 0$$

with initial data

$$u(0,x) = \begin{cases} 1, & x \le 0, \\ 0, & 0 < x. \end{cases}$$

Is the solution a classical or a weak solution?

Problem 2 Consider the quasilinear equation

$$u_t + x u_x = f \tag{1}$$

for a given function f.

a) Use the method of characteristics to solve (1) with initial data $u(0,x) = u_0(x) \in C^1(\mathbb{R})$ and $f(t,x) = 0$. Identify the largest domain on which a classical solution exists.

b) Use Duhamel's method/principle to solve (1) with initial data $u(0,x) = 0$ and $f(t,x) \in C^1(\mathbb{R}^2)$ for $t > 0$. Thereafter show that the obtained solution is indeed a solution to (1).

Problem 3 Consider the wave equation

$$u_{tt} - u_{xx} = 0, \quad (t,x) \in (0,\infty) \times (-1,1). \tag{2}$$

a) Show that if $v(t,x)$ such that $v(t,\cdot) \in H_0^1((-1,1))$ for all $t \in [0,\infty)$ satisfies

$$\int_0^\infty \int_{-1}^1 [v\psi_{tt} + v_x \psi_x](t,x)\,dx\,dt = 0 \quad \text{for all } \psi \in C^\infty_{cpt}((0,\infty) \times (-1,1)) \tag{3}$$

then also $u(t,x) = v(t,x) + a + bx$ satisfies (3).

b) A solution $u(t,x)$ to (2) is called a weak solution, if it satisfies (3) and $u(t,\cdot) \in H^1((-1,1))$ for all $t \ge 0$.

Find a candidate for a weak solution for the wave equation (2) with initial data

$$u(0,x) = 5 + x + \begin{cases} 1 + x, & x \le 0, \\ 1 - x, & 0 \le x, \end{cases} \quad \text{and} \quad u_t(0,x) = 0.$$

and boundary values

$$u(t,-1) = 4 \quad \text{and} \quad u(t,1) = 6 \quad \text{for all } t \ge 0.$$

Problem 4 Consider the heat equation on $[0, \infty) \times \mathbb{R}^n$

$$\begin{aligned} u_t - \Delta u &= f, \\ u(0, \mathbf{x}) &= \phi(\mathbf{x}) \end{aligned}$$

for given functions f and ϕ and write $\mathbf{x} = (\mathbf{y}, x_n)$, where $\mathbf{y} = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Show that if $f(t, \mathbf{y}, x_n) = f(t, \mathbf{y}, -x_n)$ and $\phi(\mathbf{y}, x_n) = \phi(\mathbf{y}, -x_n)$, then $u(t, \mathbf{y}, x_n) = u(t, \mathbf{y}, -x_n)$ for any bounded solution u.

Problem 5 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $c \in \mathbb{R}$. Assume that $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ is a solution to

$$u_t - \Delta u + cu = 0$$

and that $v \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ satisfies

$$v_t - \Delta v + cv \leq 0.$$

Show that if $v \leq u$ on Γ, the parabolic boundary, then $v \leq u$ on $\overline{\Omega_T}$.

Hint: Which equation does $w(t, \mathbf{x}) = e^{\gamma t} u(t, \mathbf{x})$ satisfy?

Problem 6 Let u be harmonic in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |u(\mathbf{x})|^2 d^n\mathbf{x} < \infty.$$

Show that $u \equiv 0$.

Problem 7 Show that the sequence $n[\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}]$ converges to $-2\delta_0'$ in $\mathcal{D}'(\mathbb{R})$, as $n \to \infty$.

Problem 8 Given $g \in C^\infty_{cpt}(\mathbb{R})$ and $u \in \mathcal{D}'(\mathbb{R})$, define

$$\nu(\phi) = u(g\phi) \quad \text{for all } \phi \in C^\infty_{\text{cpt}}(\mathbb{R}).$$

Show that $\nu \in \mathcal{D}'(\mathbb{R})$.