1) CONSIDER $u_t + u^3 u_x = 0$ $(*)$

- IF $u(0,x) = 0$ $\forall x \in \mathbb{R}$ $\Rightarrow$ $u(t,x) = 0$ $\forall (t,x)$ IS A CLASSICAL SOLUTION TO $(*)$

- IF $u(0,x) = 1$ $\forall x \in \mathbb{R}$ $\Rightarrow$ $u(t,x) = 1$ $\forall (t,x)$ IS A CLASSICAL SOLUTION TO $(*)$

$\Rightarrow$ (BY THM 10.6)
$$u(t,x) = \begin{cases} 1 & \text{IF } x < \sigma(t) \\ 0 & \text{IF } \sigma(t) < x \end{cases}$$

IS A WEAK SOLUTION TO $(*)$ IF $\sigma(t)$ SATISFIES THE RANKINE-HUGONIOT CONDITION

HERE: $u_t + u^3 u_x = 0 \Leftrightarrow u_t + \left(\frac{1}{4}u^4\right)_x = 0 \Longleftrightarrow u_t + f(u)_x = 0$ WITH $f(u) = \frac{1}{4}u^4$

$$u^-(t) = \lim_{x \uparrow \sigma(t)} u(t,x) = 1, \quad u^+(t) = \lim_{x \downarrow \sigma(t)} u(t,x) = 0$$

$$\Rightarrow \sigma'(t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = \frac{1}{4} \quad \rightsquigarrow \sigma(t) = \sigma(0) + \frac{1}{4}t$$

$$u(0,x) = \begin{cases} 1 & x \leq 0 \\ 0 & 0 < x \end{cases} \quad \Rightarrow \sigma(0) = 0 \quad \text{(JUMP LOCATION AT } t=0)$$

$$\Rightarrow \sigma(t) = \frac{1}{4}t$$

$$\Rightarrow u(t,x) = \begin{cases} 1 & x \leq \frac{1}{4}t \\ 0 & \frac{1}{4}t < x \end{cases}$$ IS A WEAK SOLUTION TO THE GIVEN IVP.

2a) CONSIDER
$$u_t + x u_x = 0$$
$$u(0,x) = u_0(x)$$

LET $z(t) = u(t,x(t))$ $\Rightarrow$ $z'(t) = u_t(t,x(t)) + x'(t) u_x(t,x(t))$

$\Rightarrow z'(t) = 0$ IF $x'(t) = x(t)$

LET $x(0) = x_0$ $\Rightarrow$ $x'(t) = x(t)$ WITH $x(0) = x_0$

$z'(t) = 0$ WITH $z(0) = u(0,x(0)) = u_0(x_0)$ $(**)$

THE UNIQUE SOLUTION TO (**) IS GIVEN BY

$$x(t) = x(0)e^{t} = x_0 e^{t} \qquad (\leadsto x_0 = x(t)e^{-t})$$

$$z(t) = z(0) = u_0(x_0)$$

BY DEFINITION: $u(t, x(t)) = z(t) = u_0(x_0) = u_0(x(t)e^{-t})$

$\Rightarrow u(t,x) = u_0(xe^{-t}) \quad \forall (t,x)$ ON THE CURVE $(t, x(t)) = (t, x_0 e^{t})$

$u(t,x) = u_0(xe^{-t})$ WILL BE A SOLUTION TO $u_t + xu_x = 0$, $u(0,x) = u_0(x)$ ON ALL OF $\mathbb{R}^2$

IF FOR ALL $(t,x) \in \mathbb{R}^2$ THERE EXISTS A UNIQUE $a \in \mathbb{R}$ ST $(t,x) = (t, ae^{t})$

LET $(t,x) \in \mathbb{R}^2 \Rightarrow x = ae^{t} \Leftrightarrow a = xe^{-t}$ AND $a \in \mathbb{R}$ UNIQUE

$\Rightarrow u(t,x) = u_0(xe^{-t})$ IS A SOLUTION TO $u_t + xu_x = 0$, $u(0,x) = u_0(x)$ ON ALL OF $\mathbb{R}^2$

2b) CONSIDER
$$\left.\begin{array}{l} u_t + xu_x = f \\ u(0,x) = 0 \quad \forall x \in \mathbb{R} \end{array}\right. \qquad (***)$$

LET $s > 0$ AND $\eta(t,x;s)$ THE SOLUTION TO
$$\eta_t(t,x;s) + x\eta_x(t,x;s) = 0 \quad (t,x) \in (s,\infty) \times \mathbb{R}$$
$$\eta(s,x;s) = f(s,x)$$

CLAIM:
$$u(t,x) = \int_0^t \eta(t,x;s)\,ds \text{ SOLVES } (***)$$

- $u(0,x) = 0$ SINCE $\eta(t,x;s) = f(s, xe^{-(t-s)})$ FROM a) AND $f \in C^1(\mathbb{R}^2)$
- $u_t(t,x) = \eta(t,x;t) + \int_0^t \eta_t(t,x;s)\,ds = f(t,x) + \int_0^t \eta_t(t,x;s)\,ds$

$$u_x(t,x) = \int_0^t \eta_x(t,x;s)\,ds$$

$$\Rightarrow u_t(t,x) + x\,u_x(t,x) = f(t,x) + \int_0^t \eta_t(t,x;s)\,ds + x\int_0^t \eta_x(t,x;s)\,ds$$

$$= f(t,x) + \int_0^t \underbrace{\eta_t(t,x;s) + x\eta_x(t,x;s)}_{=0}\,ds = f(t,x)$$

3a) LET $u(t,x) = v(t,x) + \rho(x)$ WHERE $\rho(x) = a + bx$ $(a, b \in \mathbb{R})$

$\Rightarrow \rho \in C^\infty((0,\infty)\times(-1,1))$, $a - |b| \le \rho(x) \le a + |b|$, $\rho'(x) = b$, $\rho''(x) = 0$ $\forall x \in (-1,1)$

$\Rightarrow \int_0^\infty \int_{-1}^1 \rho(x)\psi_{tt}(t,x) + \rho'(x)\psi_x(t,x)\,dx\,dt$ WELL-DEFINED AND FINITE

$\cdot)$ $\int_0^\infty \int_{-1}^1 \rho(x)\psi_{tt}(t,x)\,dx\,dt = \int_{-1}^1 \int_0^\infty \rho(x)\psi_{tt}(t,x)\,dt\,dx$ (FUBINI'S THM)

$$= -\int_{-1}^1 \rho(x)\psi_t(t,x)\Big|_{t=0}^{"\infty"} dx = 0 \qquad (\psi \in C_c^\infty((0,\infty)\times(-1,1))$$

$\cdot)$ $\int_0^\infty \int_{-1}^1 \rho'(x)\psi_x(t,x)\,dx\,dt = b\int_0^\infty \int_{-1}^1 \psi_x(t,x)\,dx\,dt$

$$= b\int_0^\infty \psi(t,x)\Big|_{x=-1}^{1} dx\,dt = 0 \qquad (\psi \in C_c^\infty((0,\infty)\times(-1,1))$$

$$\Rightarrow \int_0^\infty \int_{-1}^1 (u\psi_{tt} + u_x\psi_x)(t,x)\,dx\,dt = \int_0^\infty \int_{-1}^1 (v\psi_{tt} + v_x\psi_x)(t,x)\,dx\,dt \qquad (=0 \text{ BY ASSUMPTION})$$

$$+ \int_0^\infty \int_{-1}^1 \rho(x)\psi_{tt}(t,x) + \rho'(x)\psi_x(t,x)\,dx\,dt \qquad (=0 \text{ SEE ABOVE})$$

$$= 0$$

3b) LET $\rho(x) = 5 + x$,
$$v(0,x) = \begin{cases} 1+x & x \le 0 \\ 1-x & 0 \le x \end{cases} \Rightarrow u(0,x) = v(0,x) + \rho(x)$$

NOTE $v(0,x)$ IS PIECEWISE LINEAR AND $v(0,\pm 1) = 0$

$\Rightarrow v(0,x) \in H_0^1((-1,1))$

a) $\Rightarrow$ $u(t,x) = v(t,x) + \rho(x)$ IS A WEAK SOLUTION

IF $v(t,x)$ IS A WEAK SOLUTION TO $v_{tt} - v_{xx} = 0 \quad (t,x) \in (0,\infty) \times (-1,1)$

$$v(0,x) = \begin{cases} 1+x & x \le 0 \\ 1-x & 0 \le x \end{cases}$$

$$v_t(0,x) = 0 \quad \forall x \in (-1,1)$$

WITH $v(t,\cdot) \in H_0^1((-1,1))$ FOR ALL $t > 0$.

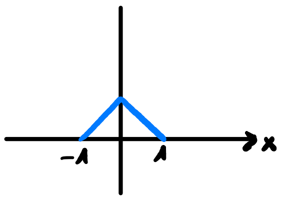
D'ALEMBERTS FORMULA: $v(t,x) = \frac{1}{2}\left(g(x+t) + g(x-t)\right)$

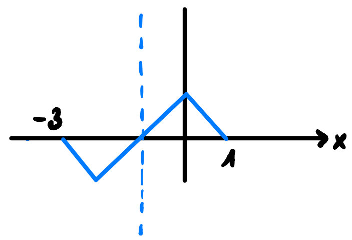
WHERE $g(x)$ IS AN EXTENSION OF $v(0,x)$ FROM $(-1,1)$ TO ALL OF $\mathbb{R}$.

- $v(t,\cdot) \in H_0^1((-1,1))$ $\forall t > 0$ $\Rightarrow$ $v(t,\pm 1) = 0$ $\forall t$ BY THM 10.9

$\Rightarrow$ $\cdot$ $0 = 2v(t,-1) = g(-1+t) + g(-1-t)$

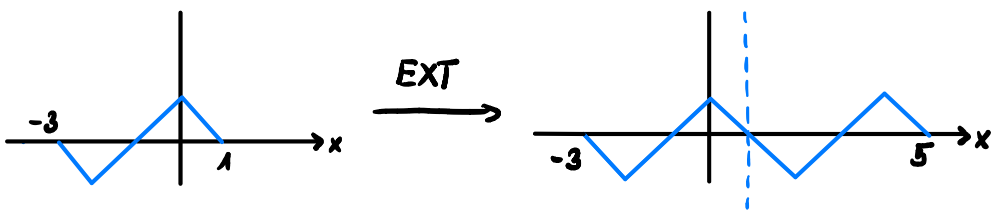
$\Rightarrow g(-1-t) = -g(-1+t)$





- $0 = 2v(t,1) = g(1+t) + g(1-t)$

$\Rightarrow g(1+t) = -g(1-t)$



USING $g(-1-t) = -g(-1+t)$ AND $g(1-t) = -g(1+t)$ AGAIN AND AGAIN

ONE OBTAINS THAT g IS THE ODD EXTENSION OF $v(0,x)$ WITH PERIOD 4.

4) LET $v(t,\vec{y},x_n) = u(t,\vec{y},-x_n)$

$\Rightarrow v_t(t,\vec{y},x_n) = u_t(t,\vec{y},-x_n)$

$v_{x_i}(t,\vec{y},x_n) = u_{x_i}(t,\vec{y},-x_n)$ AND $v_{x_ix_i}(t,\vec{y},x_n) = u_{x_ix_i}(t,\vec{y},-x_n)$ $\forall i\in\{1,2,\dots,n-1\}$

$v_{x_n}(t,\vec{y},x_n) = -u_{x_n}(t,\vec{y},-x_n)$ AND $v_{x_nx_n}(t,\vec{y},x_n) = u_{x_nx_n}(t,\vec{y},-x_n)$

$\Rightarrow (v_t-\Delta v)(t,\vec{y},x_n) = (u_t-\Delta u)(t,\vec{y},-x_n) = f(t,\vec{y},-x_n) = f(t,\vec{y},x_n)$

$v(0,\vec{y},x_n) = u(0,\vec{y},-x_n) = \phi(\vec{y},-x_n) = \phi(\vec{y},x_n)$

$\Rightarrow v$ IS A SOLUTION TO: $v_t-\Delta v = f$ ON $[0,\infty)\times\mathbb{R}^n$

$v(0,\vec{x}) = \phi(\vec{x})$

$\Rightarrow w = u-v$ IS A SOLUTION TO $w_t-\Delta w = 0$ ON $[0,\infty)\times\mathbb{R}^n$ (****)

$w(0,\vec{x}) = 0$

u BOUNDED $\Rightarrow$ v BOUNDED $\Rightarrow$ w BOUNDED

$\Rightarrow$ (****) HAS THE UNIQUE SOLUTION $w(t,\vec{x}) = 0$ $\forall(t,\vec{x})$

$\Rightarrow u(t,\vec{y},x_n) = v(t,\vec{y},x_n) = u(t,\vec{y},-x_n)$

5) LET $w = v-u$ $\Rightarrow$ $w_t-\Delta w+cw = \underbrace{(v_t-\Delta v+cv)}_{\leq 0} - \underbrace{(u_t-\Delta u+cu)}_{=0} \leq 0$

CONSIDER: $f(t,\vec{x}) = e^{ct}w(t,\vec{x})$

$\Rightarrow f_t(t,x) = e^{ct}(w_t(t,\vec{x})+cw(t,x))$

$\Delta f(t,\vec{x}) = e^{ct}\Delta w(t,\vec{x})$

$\Rightarrow f_t(t,x)-\Delta f(t,x) = \underbrace{e^{ct}}_{>0}\underbrace{(w_t-\Delta w+cw)}_{\leq 0}(t,x) \leq 0$

WEAK MAX. PRINCIPLE $\Rightarrow$ $f(t,\vec{x}) \leq \max_{\Gamma} f$ $\forall(t,\vec{x})\in\overline{\Omega_T}$

$\Rightarrow \underbrace{e^{ct}}_{\geq 0}w(t,\vec{x}) \leq \max_{\Gamma} e^{ct}w = \max_{\Gamma} \underbrace{e^{ct}}_{\geq 0}\underbrace{(v-u)}_{\leq 0} \leq 0$ $\forall(t,\vec{x})\in\overline{\Omega_T}$

$\Rightarrow (v-u)(t,\vec{x}) = w(t,\vec{x}) \leq 0 \quad \forall (t,\vec{x}) \in \Omega_T$

$\Rightarrow v(t,\vec{x}) \leq u(t,\vec{x}) \quad \forall (t,\vec{x}) \in \overline{\Omega}_T$

6) LET $\vec{x} \in \mathbb{R}^n$.

u HARMONIC ON $\mathbb{R}^n$ $\Rightarrow$ $u(\vec{x}) = \frac{1}{|B(\vec{x},R)|} \int_{B(\vec{x},R)} u(\vec{y}) d^n\vec{y} \quad \forall R$ ST $B(\vec{x},R) \subseteq \mathbb{R}^n$

[$|B(\vec{x},R)|$ ... VOLUME OF THE BALL CENTERED AT $\vec{x} \in \mathbb{R}^n$ WITH RADIUS R]

$$\overset{\text{CAUCHY}}{\underset{\text{SCHWARZ}}{\Rightarrow}} \quad |u(\vec{x})| = \frac{1}{|B(\vec{x},R)|} \left| \int_{B(\vec{x},R)} u(\vec{y}) d^n\vec{y} \right|$$

$$\leq \frac{1}{|B(\vec{x},R)|} \sqrt{|B(\vec{x},R)|} \left(\int_{B(\vec{x},R)} u^2(\vec{y}) d^n\vec{y} \right)^{1/2}$$

$$= \frac{1}{\sqrt{|B(\vec{x},R)|}} \left(\int_{B(\vec{x},R)} u^2(\vec{y}) d^n\vec{y} \right)^{1/2}$$

SINCE $B(\vec{x},R) \subseteq \mathbb{R}^n \quad \forall R>0$ AND $\int_{B(x,R)} u^2(\vec{y}) d^n\vec{y} = M < \infty$

$\Rightarrow |u(\vec{x})| \leq \frac{1}{\sqrt{|B(\vec{x},R)|}} M \to 0 \quad$ AS $R \to \infty$

$\Rightarrow u(\vec{x}) = 0$

SINCE $\vec{x} \in \mathbb{R}^n$ ARBITRARY $\Rightarrow u \equiv 0$

7) $n[\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}]$ BELONGS TO $\mathcal{D}'(\mathbb{R})$ FOR ALL $n \in \mathbb{N}$ SINCE $\mathcal{D}'(\mathbb{R})$ IS A VECTOR SPACE

$-2\delta_0'$ BELONGS TO $\mathcal{D}'(\mathbb{R})$ SINCE $\mathcal{D}'(\mathbb{R})$ IS A VECTOR SPACE AND EVERY DISTRIBUTION IS INFINITELY MANY TIMES DIFFERENTIABLE

TO SHOW: $n[\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}](\phi) \to -2\delta_0'(\phi) \quad \forall \phi \in C_\infty^c(\mathbb{R}) \quad$ AS $n \to \infty$

LET $\phi \in C_c^\infty(\mathbb{R})$:

$$n[\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}](\phi) = n[\delta_{\frac{1}{n}}(\phi) - \delta_{-\frac{1}{n}}(\phi)]$$

$$= n(\phi(\tfrac{1}{n}) - \phi(-\tfrac{1}{n}))$$

$$= 2\,\frac{\phi(\frac{1}{n}) - \phi(-\frac{1}{n})}{\frac{2}{n}} \to 2\phi'(0) \quad (n\to\infty)$$

DUE TO $\phi \in C_c^\infty(\mathbb{R}^n)$ AND THE DEF OF THE DERIVATIVE

$$-2\delta_0'(\phi) = 2\delta_0(\phi') \qquad \text{(DEF OF DISTRIBUTIONAL DERIVATIVE)}$$

$$= 2\phi'(0)$$

$\Rightarrow n[\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}](\phi) \to -2\delta_0'(\phi)$

ϕ ARBITRARY $\Rightarrow n[\delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}] \to -2\delta_0'$ IN THE SENSE OF DISTRIBUTIONS

8) •) $\nu : C_c^\infty \to \mathbb{R}$?

$g, \phi \in C_c^\infty(\mathbb{R}) \to g\phi \in C_c^\infty(\mathbb{R}) \Rightarrow u(g\phi) \in \mathbb{R}$ SINCE $u \in \mathcal{D}'(\mathbb{R})$

•) $\phi, \psi \in C_c^\infty(\mathbb{R})$

$\Rightarrow \nu(\phi+\psi) = u(g(\phi+\psi)) = u(g\phi + g\psi) \underset{u\in\mathcal{D}'(\mathbb{R})}{=} u(g\phi) + u(g\psi) = \nu(\phi) + \nu(\psi)$

•) $c \in \mathbb{R}, \phi \in C_c^\infty(\mathbb{R})$

$\Rightarrow \nu(c\phi) = u(gc\phi) = u(cg\phi) \underset{u\in\mathcal{D}'(\mathbb{R})}{=} c\,u(g\phi) = c\,\nu(\phi)$

•) ASSUME $\phi_n \to \phi$ IN $C_c^\infty(\mathbb{R})$

$\Rightarrow$ •) $\exists K$ ... COMPACT SUBSET OF $\mathbb{R}$ SUCH THAT $\operatorname{supp}(\phi_n), \operatorname{supp}(\phi) \subseteq K$ $\forall n$

•) $\sup_{x\in\mathbb{R}} |D^\alpha(\phi_n - \phi)| \to 0$ $(n\to\infty)$ $\forall$ MULTIINDEX α

$\Rightarrow$ •) $\operatorname{supp}(g\phi_n), \operatorname{supp}(g\phi) \subseteq K$ $\forall n$

•) $\sup_{x\in\mathbb{R}} |D^\alpha(g(\phi_n - \phi))| \to 0$ $(n\to\infty)$ $\forall$ MULTIINDEX α

$\Rightarrow g\phi_n \to g\phi$ IN $C_c^\infty(\mathbb{R})$

$\Rightarrow \nu(\phi_n) - \nu(\phi) = u(g\phi_n) - u(g\phi) \underset{u\in\mathcal{D}'(\mathbb{R})}{\longrightarrow} 0$ AS $n\to\infty$

$\Rightarrow \nu : \mathcal{C}_c^\infty(\mathbb{R}) \to \mathbb{R}$ IS LINEAR AND CONTINUOUS

$\Rightarrow \nu \in \mathcal{D}'(\mathbb{R})$