QUASILINEAR EQUATIONS AND CHARACTERISTICS

AIM: IDENTIFY A METHOD TO SOLVE $\vec{a}(\vec{x}, u(\vec{x}))\cdot\nabla u(\vec{x}) = c(\vec{x}, u(\vec{x}))$,

$$u(\vec{\xi}) = g(\vec{\xi}) \quad \forall \vec{\xi}\in\gamma$$

WHERE $\vec{x}\in\mathbb{R}^n$, $u: \mathbb{R}^n\to\mathbb{R}$, $\gamma\subseteq\mathbb{R}^n$ IS A HYPERSURFACE; IE A SURFACE OF DIMENSION $n-1$.
$\vec{a}: \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$, $c: \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$

MOTIVATING EXAMPLE:

$$\begin{aligned} &u_t + b u_x = 0 \quad b\in\mathbb{R} \\ &u(0,x) = g(x)\in C^1(\mathbb{R}) \end{aligned} \qquad (*)$$

$u(t,x) = g(x-bt)$ IS A SOLUTION TO $(*)$, BUT HOW CAN IT BE COMPUTED?

LET $S = \{(t,x,u(t,x))\in\mathbb{R}^3 \mid (t,x)\in\mathbb{R}^2\}$ ... GRAPH OF u

CONSIDER A CURVE $\Gamma = \{(t, x(t), z(t)) \mid t\in\mathbb{R}\}\subseteq S$ IN $\mathbb{R}^3$

$\Rightarrow z(t) = u(t, x(t))$

$\Rightarrow z'(t) = u_t(t,x(t)) + u_x(t,x(t))x'(t)$

$\Rightarrow z'(t) = 0$ IF $x'(t) = b$

$\Rightarrow$ ALONG Γ $(*)$ IS SATISFIED IF $\begin{aligned} z'(t) &= 0 \\ x'(t) &= b \end{aligned}$ $(**)$

$\Rightarrow$ IF $(0, x_0, u(0,x_0)) = (0, x_0, g(x_0))\in\Gamma$, THEN $\begin{aligned} z(t) &= z(0) = g(x_0) \\ x(t) &= x_0 + bt \end{aligned}$

IN PARTICULAR $u(t, x(t)) = z(t) = g(x_0) = g(x(t) - bt)$

$\Rightarrow u(t,x) = g(x-bt)$ ALONG THE LINE $x(t) = x_0 + bt$

NOTE: • SINCE $x_0\in\mathbb{R}$ ARBITRARY AND $\mathbb{R}^2 = \bigcup_{x_0\in\mathbb{R}} \{(t, x_0+bt) \mid t\in\mathbb{R}\}$

$\Rightarrow$ THE SOLUTION CAN BE COMPUTED ON ALL OF $\mathbb{R}^2$
EACH POINT (t,x) LIES ON EXACTLY ONE LINE $x(t) = x_0 + bt$

• $g\in C^1(\mathbb{R})$ ENSURES $u\in C^1(\mathbb{R}^2;\mathbb{R})$ AND ALL DERIVATIVES THAT TURN UP ARE WELL-DEFINED

a) CONSIDER:

$$\begin{aligned} &u_t + v u_x = w \\ &u(0,x) = g(x)\in C^1(\mathbb{R}) \end{aligned} \qquad [v(t,x), w(t,x,u(t,x))] \qquad (**)$$

TRY THE ANSATZ $z(t) = u(t,x(t)) \Rightarrow z'(t) = u_t(t,x(t)) + u_x(t,x(t))x'(t)$

$\Rightarrow z'(t) = w(t, x(t), u(t,x(t))) = w(t, x(t), z(t))$ IF $x'(t) = v(t, x(t))$

$\Rightarrow$ THE PDE REWRITES AS

$$\begin{aligned} z'(t) &= w(t, x(t), z(t)) \\ x'(t) &= v(t, x(t)) \end{aligned} \qquad (***)$$

ALONG THE CHARACTERISTIC $x(t)$.

ISSUES: -) CAN (xxx) BE UNIQUELY SOLVED FOR A GIVEN INITIAL DATA

$(x(0), z(0)) = (x(0), u(0, x(0))) = (x(0), g(x(0)))$?

ODE THEORY: YES, IF ·) $v(t,x), v_x(t,x)$ ARE CONTINUOUS

·) $w(t,x,z), w_x(t,x,z), w_z(t,x,z)$ ARE CONTINUOUS

-) DOES (xxx) HAVE A LOCAL OR GLOBAL SOLUTION ?

-) LET $x(t;x_0)$ DENOTE THE CHARACTERISTIC $x(t)$ SUCH THAT $x(t) = x_0$.

IS $\bigcup_{x_0 \in \mathbb{R}} \{(t, x(t;x_0)) \mid t \in \mathbb{R}\} = \mathbb{R}^2$?

IF, SO DOES EACH POINT $(t,x) \in \mathbb{R}^2$ LIE ON EXACTLY ONE CHARACTERISTIC ?

EXAMPLE: $u_t - x u_x = -2u$

$u(0,x) = x$

$\Rightarrow$ MUST SOLVE $\begin{array}{l} \dot{x}(t) = -x(t) \\ \dot{z}(t) = -2u(t,x(t)) = -2z(t) \end{array}$ WITH INITIAL DATA $\begin{array}{l} x(0) = x_0 \\ z(0) = u(0,x_0) = x_0 \end{array}$

·) $\dot{x}(t) = -x(t) \Rightarrow x(t) = x_0 e^{-t}$

·) $\dot{z}(t) = -2z(t) \Rightarrow z(t) = z(0)e^{-2t} = x_0 e^{-2t}$

$\Rightarrow u(t,x(t)) = z(t) = x_0 e^{-2t} = x(t) e^{-t}$

$\Rightarrow u(t,x) = x e^{-t}$

LET $x(t;x_0)$... CHARACTERISTIC $x(t)$ SUCH THAT $x(t) = x_0$.

GIVEN $(t,x) \in \mathbb{R}^2$, DOES THERE EXIST $x_0 \in \mathbb{R}$ UNIQUE SUCH THAT $(t,x) = (t, x(t;x_0))$?

$x = x(t;x_0) \Leftrightarrow x = x_0 e^{-2t} \Leftrightarrow x_0 = x e^{+2t}$, SO YES !

EXAMPLE: $u_t - x u_x = u^2$

$u(0,x) = 1$

$\Rightarrow$ MUST SOLVE $\begin{array}{l} \dot{x}(t) = -x(t) \\ \dot{z}(t) = z^2(t) \end{array}$ WITH INITIAL DATA $\begin{array}{l} x(0) = x_0 \\ z(0) = 1 \end{array}$

·) $\dot{x}(t) = -x(t) \Rightarrow x(t) = x_0 e^{-t}$

·) $\dot{z}(t) = z^2(t) \Rightarrow z(t) = \frac{z(0)}{1 - z(0)t} = \frac{1}{1-t}$ $\Rightarrow u(t,x(t)) = \frac{1}{1-t}$ $\Rightarrow u(t,x) = \frac{1}{1-t}$

$\rightsquigarrow$ $z(t)$ AND $u(t,x)$ ONLY WELL DEFINED FOR $t < 1$!

$\Rightarrow$ NO SOLUTION ON ALL OF $\mathbb{R}^2$, BUT ON $(-\infty, 1) \times \mathbb{R}$!

b) CONSIDER $u_t + a(u) u_x = 0$ $[a \in C^1(\mathbb{R})]$

$u(0,x) = g(x) \in C^1(\mathbb{R})$

TRY THE ANSATZ $z(t) = u(t,x(t)) \Rightarrow \dot{z}(t) = u_t(t,x(t)) + u_x(t,x(t))\dot{x}(t)$

$\Rightarrow \dot{z}(t) = 0$ IF $\dot{x}(t) = a(u(t,x(t))) = a(z(t))$

$\Rightarrow$ THE PDE REWRITES AS $\begin{array}{l} \dot{z}(t) = 0 \\ \dot{x}(t) = a(z(t)) \end{array}$ (xxxx)

ALONG THE CHARACTERISTIC $x(t)$

ISSUES: $\to$ CAN (****) BE UNIQUELY SOLVED FOR A GIVEN INITIAL DATA $(x(0),z(0))=(x(0),g(x(0)))$?

ODE THEORY: YES IF $\cdot)$ a, a_u ARE CONTINUOUS

[$x'(t)=a(u(t,x(t)))$ FOR u KNOWN, REQUIRES IN ADDITION u, u_x CONTINUOUS]

$\to$ DOES (****) HAVE A LOCAL OR GLOBAL SOLUTION?

$\to$ LET $x(t;x_0)$ DENOTE THE CHARACTERISTIC $x(t)$ SUCH THAT $x(t)=x_0$.

IS $\bigcup_{x_0\in\mathbb{R}} \{(t,x(t;x_0)) \mid t\in I_{x_0}\} = \Omega$?

IF SO DOES EACH POINT $(t,x)\in\Omega$ LIE ON EXACTLY ONE CHARACTERISTIC?

EXAMPLE: $u_t+(1-2u)u_x=0$

$u(0,x)=\frac{1}{2}-\frac{1}{\pi}\arctan(x)$ $\quad [(t,x)\in[0,\infty)\times\mathbb{R}]$

$\Rightarrow$ MUST SOLVE $x'(t)=1-2z(t)$, $z'(t)=0$ WITH INITIAL DATA $x(0)=x_0$, $z(0)=\frac{1}{2}-\frac{1}{\pi}\arctan(x_0)$

$\cdot)$ $z(t)=z(0)=\frac{1}{2}-\frac{1}{\pi}\arctan(x_0)$

$\cdot)$ $x'(t)=1-2z(t)=1-2z(0) \Rightarrow x(t)=x_0+(1-2z(0))t=x_0+\frac{1}{\pi}\arctan(x_0)t$

IMPOSSIBLE TO EXPRESS x_0 EXPLICITLY WITH THE HELP OF x & t.

WHICH IS NEEDED SINCE

$u(t,x(t))=z(t)=\frac{1}{2}-\frac{1}{\pi}\arctan(x_0)$ $\qquad$ (*****)

WHAT CAN WE SAY ABOUT THE SOLUTION?

LET $x(t;x_0)$... CHARACTERISTIC $x(t)$ SUCH THAT $x(0)=x_0$

$\Rightarrow x(t;x_0)=x_0+\frac{1}{\pi}\arctan(x_0)t$

IF FOR EACH $t\in\mathbb{R}^+$: $\{x(t;x_0) \mid x_0\in\mathbb{R}\}=\mathbb{R}$

$\Rightarrow \bigcup_{t\in\mathbb{R}} \{(t,x(t;x_0)) \mid x_0\in\mathbb{R}\}=[0,\infty)\times\mathbb{R}=\bigcup_{x_0\in\mathbb{R}} \{(t,x(t;x_0)) \mid x_0\in\mathbb{R}\}$

IF, IN ADDITION, FOR EACH $t\in\mathbb{R}^+$: $x(t,\cdot): \mathbb{R}\to\mathbb{R}$, $x_0\mapsto x(t;x_0)$ IS STRICTLY INCREASING

$\Rightarrow$ EACH POINT $(t,x)\in[0,\infty)\times\mathbb{R}$ LIES ON EXACTLY ONE CHARACTERISTIC.

FOR EACH $t\in\mathbb{R}^+$: $\lim_{x_0\to\pm\infty} x(t;x_0)=\pm\infty$, $x_{x_0}(t;x_0)=1+\frac{1}{\pi}\frac{1}{1+x_0^2}>0$

$\Rightarrow$ YES, EACH POINT $(t,x)\in[0,\infty)\times\mathbb{R}$ LIES ON EXACTLY ONE CHARACTERISTIC

$\Rightarrow u(t,x)$ IS IMPLICITELY GIVEN BY (*****) ON ALL OF $[0,\infty)\times\mathbb{R}$.

EXAMPLE: $u_t + (1-2u)u_x = 0$ $\quad [(t,x) \in [0,\infty) \times \mathbb{R}]$

$u(0,x) = \arctan(x)$

$\Rightarrow$ TRY TO SOLVE: $x'(t) = 1 - 2z(t)$, $z'(t) = 0$ WITH INITIAL DATA $x(0) = x_0$, $z(0) = \arctan(x_0)$

- $z(t) = z(0) = \arctan(x_0)$
- $x'(t) = 1 - 2z(t) = 1 - 2z(0) \Rightarrow x(t) = x_0 + (1 - 2\arctan(x_0))t$

IMPOSSIBLE TO EXPRESS x_0 EXPLICITLY WITH THE HELP OF x & t

AS BEFORE: $x(t; x_0)$ ... CHARACTERISTIC $x(t)$ SUCH THAT $x(0) = x_0$

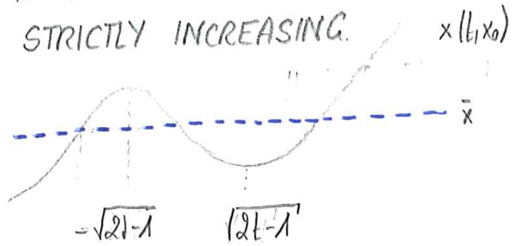
$\rightarrow x(t; x_0) = x_0 + (1 - 2\arctan(x_0))t$

FOR EACH $t \in \mathbb{R}^+$: $\lim_{x_0 \to \pm\infty} x(t; x_0) = \pm\infty$,

$$x_{x_0}(t; x_0) = 1 - \frac{2t}{1+x_0^2} = \frac{1 + x_0^2 - 2t}{1 + x_0^2}$$

NOTE THAT $x_{x_0}(t; x_0) = 0 \Leftrightarrow 1 + x_0^2 = 2t$

$\Rightarrow$ FOR $t > \frac{1}{2}$ THE MAPPING $x_0 \mapsto x(t; x_0)$ IS NO LONGER STRICTLY INCREASING.



$\Rightarrow$ THREE POSSIBLE VALUES OF $u(t, \bar{x})$!

$\Rightarrow$ THE SOLUTION $u(t,x)$ IS ONLY WELL-DEFINED FOR $(t,x) \in [0, \frac{1}{2}] \times \mathbb{R}$!

c) GENERAL QUASILINEAR EQUATION IN TWO DIMENSIONS:

$$a u_x + b u_y = c \qquad (\text{xx})$$

$u: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a(x,y,u(x,y))$, $b(x,y,u(x,y))$, $c(x,y,u(x,y))$

CONSIDER ANY SMOOTH CURVE $\Gamma = \{(x(\tau), y(\tau)) \mid \tau \in I\}$

LET $z(\tau) = u(x(\tau), y(\tau)) \Rightarrow z'(\tau) = u_x(x(\tau), y(\tau))x'(\tau) + u_y(x(\tau), y(\tau))y'(\tau)$

$\Rightarrow z'(\tau) = c(x(\tau), y(\tau), u(x(\tau), y(\tau))) = c(x(\tau), y(\tau), z(\tau))$

IF $x'(\tau) = a(x(\tau), y(\tau), z(\tau))$
$y'(\tau) = b(x(\tau), y(\tau), z(\tau))$

$\Rightarrow$ THE PDE REWRITES AS

$$\begin{aligned} z'(\tau) &= c(x(\tau), y(\tau), z(\tau)) \\ x'(\tau) &= a(x(\tau), y(\tau), z(\tau)) \qquad (\text{x}) \\ y'(\tau) &= b(x(\tau), y(\tau), z(\tau)) \end{aligned}$$

ALONG THE CHARACTERISTIC $(x(\tau), y(\tau))$

ISSUES: -) (x) CAN BE UNIQUELY SOLVED (LOCALLY), IF
$a, \nabla a, b, \nabla b, c, \nabla c$ ARE CONTINUOUS

-) (x) MUST NOT HAVE A GLOBAL SOLUTION

-) TO TAKE THE STEP FROM ONE SINGLE CURVE TO A SET OF CHARACTERISTICS COVERING A DOMAIN Ω:

LOOK AT CAUCHY PROBLEM:

$$a u_x + b u_y = c$$
$u = g$ on γ... A SMOOTH CURVE IN $\mathbb{R}^2$

LET $\gamma = \{ \vec{r}(\sigma) = (r_1(\sigma), r_2(\sigma)) \mid \sigma \in I \}$ AND DEFINE $(x(\tau,\sigma), y(\tau,\sigma))$ TO BE THE CHARACTERISTIC $(x(\tau), y(\tau))$ SUCH THAT $(x(0), y(0)) = (r_1(\sigma), r_2(\sigma))$

IS $\bigcup_{\sigma \in I} \{(x(\tau,\sigma), y(\tau,\sigma)) \mid \tau \in I_\sigma\} = \Omega$?

IF SO, DOES EACH POINT $(x,y) \in \Omega$ LIE ON EXACTLY ONE CHARACTERISTIC?

IF SO, THEN $(x,y): \Lambda \to \Omega$
$(\sigma,\tau) \mapsto (x(\sigma,\tau), y(\sigma,\tau))$ IS INVERTIBLE (EVEN C^1)

$$\Rightarrow \begin{pmatrix} x_\sigma & x_\tau \\ y_\sigma & y_\tau \end{pmatrix} \text{ IS NON-SINGULAR. (INVERTIBLE)} \qquad (xxx)$$

AND u IS DEFINED BY
$$u(x(\sigma,\tau), y(\sigma,\tau)) = z(\sigma,\tau)...$$

NOTE (xxx) AT $\tau = 0$ REQUIRES γ TO NOT BE TANGENT TO ANY CHARACTERISTIC CURVE!

EXAMPLE: $u u_x + x u_y = 0$
$u(0,s) = 2s$ FOR $s > 0$

HERE: $\gamma = \{(0,\sigma) \mid \sigma > 0\} \Rightarrow (x(0,\sigma), y(0,\sigma)) = (0,\sigma) \quad (\sigma > 0) \Rightarrow z(0,\sigma) = u(x(0,\sigma), y(0,\sigma)) = 2\sigma$

$$x_\tau(\tau,\sigma) = u(x(\tau,\sigma), y(\tau,\sigma)) = z(\tau,\sigma)$$
$$y_\tau(\tau,\sigma) = x(\tau,\sigma)$$
$$z_\tau(\tau,\sigma) = 0$$

$$\Rightarrow z(\tau,\sigma) = z(0,\sigma) = 2\sigma$$
$$x(\tau,\sigma) = 2\sigma\tau$$
$$y(\tau,\sigma) = \sigma + \sigma\tau^2 = \sigma(1+\tau^2)$$

$$\begin{pmatrix} x_\tau & x_\sigma \\ y_\tau & y_\sigma \end{pmatrix} = \begin{pmatrix} 2\sigma & 2\tau \\ 2\sigma\tau & 1+\tau^2 \end{pmatrix} \Rightarrow \begin{vmatrix} x_\tau & x_\sigma \\ y_\tau & y_\sigma \end{vmatrix} = 2\sigma(1-\tau^2)$$

ONLY NON-SINGULAR FOR $\tau \notin \{-1,1\}$!

$$\Rightarrow u(x(\tau,\sigma), y(\tau,\sigma)) = z(\tau,\sigma) = 2\sigma$$

$$y(\tau,\sigma) = \sigma(1+\tau^2) = \sigma + \frac{1}{4\sigma} x(\tau,\sigma)^2 \Rightarrow \sigma^2 - \sigma y(\tau,\sigma) + \frac{1}{4} x(\tau,\sigma)^2 = 0$$

$$\sigma_{1,2} = \frac{y(\tau,\sigma)}{2} \pm \sqrt{\frac{y(\tau,\sigma)^2}{4} - \frac{1}{4} x(\tau,\sigma)^2}$$

SINCE $u(x(0,\sigma), y(0,\sigma)) = u(0,\sigma) = 2\sigma \Rightarrow 2\sigma = 2y(0,\sigma) = y(0,\sigma) + \sqrt{y(0,\sigma)^2 - x(0,\sigma)^2}$

$$\Rightarrow u(x,y) = y + \sqrt{y^2 - x^2} \text{ .. VALID FOR } y > |x|.$$

d) THE GENERAL QUASILINEAR EQUATION IN HIGHER DIMENSIONS:

$$\vec{a}(\vec{x}, u(\vec{x})) \cdot \nabla u(\vec{x}) = c(\vec{x}, u(\vec{x})) \qquad \vec{x} \in \mathbb{R}^n$$
$$u(\vec{\xi}) = g(\vec{\xi}) \qquad \vec{\xi} \in \gamma$$

WHERE $\gamma \subseteq \mathbb{R}^n$ IS A HYPERSURFACE; IE A SURFACE OF DIMENSION $n-1$.

THE CHARACTERISTIC EQUATIONS BECOME

$$\vec{x}_\tau(\tau, \vec{\xi}) = \vec{a}(\vec{x}(\tau, \vec{\xi}), z(\tau, \vec{\xi})) \quad \text{WHERE} \quad z(\tau, \vec{\xi}) = u(\vec{x}(\tau, \vec{\xi}))$$
$$z_\tau(\tau, \vec{\xi}) = c(\vec{x}(\tau, \vec{\xi}), z(\tau, \vec{\xi}))$$

WITH INITIAL CONDITION $\vec{x}(0, \vec{\xi}) = \vec{\xi}$
$$z(0, \vec{\xi}) = g(\vec{\xi})$$

AGAIN $u(\vec{x}(\tau, \vec{\xi})) = z(\tau, \vec{\xi})$ TO BE SOLVABLE REQUIRES $\vec{x}$ TO BE INVERTIBLE; [illegible]

[illegible] CRITERIA FOR $\vec{x}$ TO BE INVERTIBLE:

[illegible] INVERSE FCT THM!

IN PARTICULAR

$$\vec{a}(\vec{\xi}, g(\vec{\xi})) = \vec{a}(\vec{x}(0, \vec{\xi}), z(0, \vec{\xi}))$$

CANNOT BE TANGENT TO γ FOR ANY $\vec{\xi} \in \gamma$