12.1-3 DISTRIBUTIONS.

MOTIVATION: CONSERVATION LAWS $\Leftrightarrow$ RANKINE HUGONIOT.

$$u_t + (f(u))_x = 0$$

u IS A WEAK SOLUTION IF $\int_0^\infty \int_{\mathbb{R}} u\phi_t + f(u)\phi_x \, dx\, dt = 0 \quad \forall \phi \in C_c^\infty((0,\infty)\times\mathbb{R})$

$\rightsquigarrow \quad u(t,x) = \begin{cases} u_-(t,x) & x < \sigma(t) \\ u_+(t,x) & \sigma(t) < x \end{cases}$; WHERE $u_\pm$ ARE CLASSICAL SOLUTIONS AND $\dot\sigma(t) = \frac{f(u_+(t,\sigma(t))) - f(u_-(t,\sigma(t)))}{u_+(t,\sigma(t)) - u_-(t,\sigma(t))}$

BUT u IS NOT WEAKLY DIFFERENTIABLE.

$\rightsquigarrow$ MUST INTRODUCE THE CONCEPT OF DISTRIBUTIONS, WHICH GENERALIZES WEAK DIFFERENTIABILITY

DEF: LET $\Omega \subseteq \mathbb{R}^n$ BE A DOMAIN. SUPPOSE $\{\phi_j\}$ IS A SEQUENCE IN $C_c^\infty(\Omega)$ AND $\phi \in C_c^\infty(\Omega)$. WE SAY THAT $\lim_{j\to\infty} \phi_j = \phi$ IF

- THERE IS A COMPACT SET $K \subseteq \Omega$ ST $\phi_j \equiv 0$ OUTSIDE K FOR ALL j.
- ALL DERIVATIVES OF ϕ_j CONVERGE UNIFORMLY TO THE CORRESPONDING DERIVATIVES OF ϕ, I.E. FOR ALL MULTIINDICES α, $\sup_\Omega |D^\alpha(\phi_j - \phi)| \to 0$ AS $j\to\infty$

DEF: LET $\Omega \subseteq \mathbb{R}^n$ BE A DOMAIN. A DISTRIBUTION $u \in \mathcal{D}'(\Omega)$ IS A CONTINUOUS LINEAR FUNCTIONAL ON $C_c^\infty(\Omega)$. THAT IS, A DISTRIBUTION u IS A MAP $u: C_c^\infty(\Omega) \to \mathbb{R}$ SUCH THAT

- u IS LINEAR: $u(\phi+\psi) = u(\phi) + u(\psi) \quad \forall \phi,\psi \in C_c^\infty(\Omega)$
  $u(c\phi) = cu(\phi) \quad \forall c \in \mathbb{R}, \forall \phi \in C_c^\infty(\Omega)$
- u IS CONTINUOUS: IF ϕ_j IS ANY SEQUENCE ST $\phi_j \to \phi$ IN $C_c^\infty(\Omega)$, THEN
  $u(\phi_j) \to u(\phi)$ IN $\mathbb{R}$.

REMARK: LET $(c_1u_1 + c_2u_2)(\phi) = c_1u_1(\phi) + c_2u_2(\phi) \quad \forall c_1,c_2 \in \mathbb{R}, u_1,u_2 \in \mathcal{D}'(\Omega), \phi \in C_c^\infty(\Omega) \Rightarrow \mathcal{D}'(\Omega)$ IS A VECTOR SPACE!

EX:
- DELTA DISTRIBUTION $\delta_{\vec a}(\phi) = \phi(\vec a)$ FOR $\vec a \in \Omega$.
- $a,b \in \Omega$: $u(\phi) = \phi_{x_1}(a) - \phi_{x_2x_2}(b)$
  - $\phi,\psi \in C_c^\infty(\Omega)$:
    $u(\phi+\psi) = (\phi+\psi)_{x_1}(a) - (\phi+\psi)_{x_2x_2}(b) = \phi_{x_1}(a) - \phi_{x_2x_2}(b) + \psi_{x_1}(a) - \psi_{x_2x_2}(b) = u(\phi) + u(\psi)$
  - $c \in \mathbb{R}, \phi \in C_c^\infty(\Omega)$
    $u(c\phi) = (c\phi)_{x_1}(a) - (c\phi)_{x_2x_2}(b) = c\phi_{x_1}(a) - c\phi_{x_2x_2}(b) = cu(\phi)$
  - IF $\phi_j \to \phi$ IN $C_c^\infty(\Omega) \Rightarrow u(\phi_j) - u(\phi) = u(\phi_j - \phi)$

$$\Rightarrow |u(\phi_j) - u(\phi)| \le \underbrace{|(\phi_j-\phi)_{x_1}(a)|}_{\to 0 \ (j\to\infty)} + \underbrace{|(\phi_j-\phi)_{x_2x_2}(b)|}_{\to 0 \ (j\to\infty)}$$

$$\Rightarrow u(\phi_j) \to u(\phi) \text{ AS } j\to\infty$$

•) $u(\phi) = \int_\Omega \phi^2(x)\,dx$ •) $c \in \mathbb{R}$, $\phi \in C_c^\infty(\Omega)$: $u(c\phi) = \int_\Omega (c\phi)^2(x)\,dx = c^2 \int_\Omega \phi^2(x)\,dx = c^2 u(\phi)$

$\Rightarrow$ NO DISTRIBUTION.

•) $f \in C(\Omega) \rightsquigarrow u(\phi) = \int_\Omega f(x)\phi(x)\,d^n\vec{x} \quad \forall \phi \in C_c^\infty(\Omega)$

•) u IS LINEAR: PROPERTIES OF THE INTEGRAL

•) u CONTINUOUS:

ASSUME $\phi_j \to \phi$ IN $C_c^\infty(\Omega)$

$\Rightarrow \exists\, K \subseteq \Omega$ COMPACT ST $\phi_j \equiv 0$ OUTSIDE K

$\Rightarrow$ •) $\phi \equiv 0$ OUTSIDE K

•) f IS BOUNDED ON K.

$$\Rightarrow |u(\phi_j) - u(\phi)| = \left| \int_\Omega f(\vec{x})(\phi_j - \phi)(\vec{x})\,d^n\vec{x} \right| = \left| \int_K f(\vec{x})(\phi_j - \phi)(\vec{x})\,d^n\vec{x} \right|$$

$$\leq \sup_{\vec{x}\in\Omega} |(\phi_j - \phi)(\vec{x})| \max_{\vec{x}\in K} |f(\vec{x})|\, \text{meas}(K) \to 0 \quad (j \to \infty)$$

$\Rightarrow u(\phi_j) \to u(\phi)$ AS $j \to \infty$

•) $f \in L^1_{loc}(\Omega) \rightsquigarrow u(\phi) = \int_\Omega f(x)\phi(x)\,d^n\vec{x} \quad \forall \phi \in C_c^\infty(\Omega)$ (TRY YOURSELF)

LEMMA: LET $\Omega \subseteq \mathbb{R}^n$ BE A DOMAIN. THE MAPPING $i: L^1_{loc}(\Omega) \to \mathcal{D}'(\Omega)$, WHERE $f \to u_f$

$u_f(\phi) = \int_\Omega f(\vec{x})\phi(\vec{x})\,d^n x$, IS INJECTIVE.

PROOF $i(f) = u_f$ IS A DISTRIBUTION: TRY YOURSELF

i IS INJECTIVE: TO SHOW IF $i(f) = i(g) \Rightarrow f = g$

$[\rightsquigarrow 0 = i(f) - i(g) = i(f-g) \Rightarrow f - g = 0]$

EQUIVALENT TO: $i(h) = 0 \Rightarrow h = 0$.

IF $i(h) = 0 \Rightarrow \int_\Omega h(\vec{x})\phi(\vec{x})\,d^n\vec{x} = 0 \quad \forall \phi \in C_c^\infty(\Omega)$

$\underset{\text{lo 1}}{\Rightarrow} h(\vec{x}) \equiv 0$

DEF: LET $\Omega \subseteq \mathbb{R}^n$ BE A DOMAIN. ONE SAYS THAT A SEQUENCE $u_j \in \mathcal{D}'(\Omega)$ CONVERGES TO $u \in \mathcal{D}'(\Omega)$ IF $u(\phi) = \lim_{j\to\infty} u_j(\phi) \quad \forall \phi \in C_c^\infty(\Omega)$

EX: •) LET $f \in L^1(\mathbb{R}^n)$ WITH COMPACT SUPPORT AND $\int_{\mathbb{R}^n} f(\vec{x})\,d^n\vec{x} = 1$. DEFINE $f_a(\vec{x}) = a^n f(a\vec{x})$ $(a > 0)$

CLAIM: $\lim_{a\to\infty} u_{f_a} = \delta_{\vec{0}}$ IN $\mathcal{D}'(\mathbb{R}^n)$

PROOF: LET $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\Rightarrow \mu_{f_a}(\phi) = \int_{\mathbb{R}^n} f_a(\vec{x})\phi(\vec{x})d^n\vec{x} = \int_{\mathbb{R}^n} a^n f(a\vec{x})\phi(\vec{x})d^n\vec{x} = \int_{\mathbb{R}^n} f(\vec{y})\phi(\tfrac{1}{a}\vec{y})d^n\vec{y} \qquad [\vec{y} = a\vec{x}]$$

$$\Rightarrow \mu_{f_a}(\phi) - \delta_{\vec{0}}(\phi) = \int_{\mathbb{R}^n} f(\vec{y})\phi(\tfrac{1}{a}\vec{y})d^n\vec{y} - \phi(\vec{0})$$

$$= \int_{\mathbb{R}^n} f(\vec{y})\left(\phi(\tfrac{1}{a}\vec{y}) - \phi(\vec{0})\right)d^n\vec{y}$$

$\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \Rightarrow$ •) $\exists M>0$ ST $|\phi(\vec{x})| \leq M \quad \forall \vec{x} \in \mathbb{R}^n \Rightarrow |f(\vec{y})(\phi(\tfrac{1}{a}\vec{y}) - \phi(\vec{0}))| \leq 2M|f(\vec{y})| \in L^1(\mathbb{R}^n)$

•) $\phi(\tfrac{1}{a}\vec{y}) \to \phi(\vec{0})$ AS $a \to \infty$

$\Rightarrow$ (DOMINATED CONVERGENCE THM)

$$\mu_{f_a}(\phi) - \delta_{\vec{0}}(\phi) = \int_{\mathbb{R}^n} f(\vec{y})\left(\phi(\tfrac{1}{a}\vec{y}) - \phi(\vec{0})\right)d^n\vec{y} \to 0 \quad (a \to \infty).$$

$\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ ARBITRARY $\Rightarrow \lim_{a\to\infty} \mu_{f_a} = \delta_{\vec{0}}$

•) HEAT KERNEL:

$$H_t(\vec{x}) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|\vec{x}\|^2}{4t}} = (4\pi t)^{-n/2} e^{-\frac{1}{4}\|\frac{1}{t}\vec{x}\|^2} \text{ SATISFIES: } \int_{\mathbb{R}^n} H_t(\vec{x})d^n\vec{x} = 1 \quad \forall t>0$$

$\Rightarrow \lim_{t\to 0} H_t = \delta_{\vec{0}}$ IN THE SENSE OF DISTRIBUTIONS

DEF: LET $\Omega \subseteq \mathbb{R}^n$ BE A DOMAIN. THE DISTRIBUTIONAL DERIVATIVES OF $\mu \in \mathcal{D}'(\Omega)$ ARE DEFINED BY

$$(D^\alpha \mu)(\phi) = (-1)^{|\alpha|}\mu(D^\alpha\phi) \qquad (*)$$

WHERE $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ AND $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$

NOTE IF $\mu \in \mathcal{D}'(\Omega)$, THEN THE RHS OF $(*)$ IS DEFINED FOR ALL MULTIINDICES α SINCE $\phi \in \mathcal{C}_c^\infty(\Omega)$

$\Rightarrow$ NO RESTRICTION TO THE ORDER OF α ?

LET α ... MULTIINDEX: •) $\phi, \psi \in \mathcal{C}_c^\infty(\Omega)$

$$(D^\alpha\mu)(\phi+\psi) = (-1)^{|\alpha|}\mu(D^\alpha(\phi+\psi)) = (-1)^\alpha(\mu(D^\alpha\phi) + \mu(D^\alpha\psi)) = (D^\alpha\mu)(\phi) - (D^\alpha\mu)(\psi)$$

•) $c \in \mathbb{R}, \phi \in \mathcal{C}_c^\infty(\Omega)$

$$(D^\alpha\mu)(c\phi) = (-1)^{|\alpha|}\mu(D^\alpha(c\phi)) = (-1)^{|\alpha|}c\,\mu(D^\alpha\phi) = c(D^\alpha\mu)(\phi)$$

•) $\phi_j \to \phi$ IN $\mathcal{C}_c^\infty(\Omega) \Rightarrow D^\alpha\phi_j \to D^\alpha\phi$ IN $\mathcal{C}_c^\infty(\Omega)$

$$\Rightarrow (D^\alpha\mu)(\phi_j) - (D^\alpha\mu)(\phi) = (-1)^{|\alpha|}\mu(D^\alpha\phi_j) - (-1)^{|\alpha|}\mu(D^\alpha\phi)$$

$$= (-1)^{|\alpha|}\mu(D^\alpha(\phi_j - \phi)) \to 0$$

$\Rightarrow (D^\alpha\mu) \in \mathcal{D}'(\Omega) \quad \forall$ MULTIINDICES α.

$\Rightarrow$ DISTRIBUTIONS ARE INFINITELY MANY TIMES DIFFERENTIABLE!

EX. •) LET $u = \delta_a \in \mathcal{D}'(\mathbb{R})$ $(a \in \mathbb{R})$ $\Rightarrow$ $u^{(n)}(\phi) := (-1)^n u(\phi^{(n)}) = (-1)^n \phi^{(n)}(a)$ $\quad \forall \phi \in C_c^\infty(\mathbb{R})$

•) LET

$$H(x) = \begin{cases} 0 & x<0 \\ 1 & 0<x \end{cases} \quad \ldots \text{ HEAVISIDE FCT}$$

$$\Rightarrow u_H \in \mathcal{D}'(\mathbb{R}) \Rightarrow u_H'(\phi) = (-1) u_H(\phi') = (-1)\int_0^\infty \phi'(y)\,dy = \phi(0) = \delta_0(\phi) \quad \forall \phi \in C_c^\infty(\mathbb{R})$$

$$\Rightarrow u_H' = \delta_0 \quad (\text{OR } H' = \delta_0).$$

•) LET $w \in L^1_{loc}(\mathbb{R})$ ST

$$w(x) = \begin{cases} w_-(x) & x<0 \\ w_+(x) & 0<x \end{cases}, \quad w_\pm \in C^1(\mathbb{R})$$

$\Rightarrow u_w \in \mathcal{D}'(\mathbb{R})$. AND

$$(u_w')(\phi) = (-1)u_w(\phi') = -\int_{\mathbb{R}} w\phi'(x)\,dx = -\int_{-\infty}^0 w_-\phi'(x)\,dx - \int_0^\infty w_+\phi'(x)\,dx$$

$$= -(w_-(0) - w_+(0))\,\phi(0) + \int_{-\infty}^0 w_-'\phi(x)\,dx + \int_0^\infty w_+'\phi(x)\,dx$$

$$= -(w_-(0) - w_+(0))\,\delta_0(\phi) + u_h(\phi)$$

WHERE

$$h(x) = \begin{cases} w_-'(x) & x<0 \\ w_+'(x) & 0<x \end{cases}$$

$\Rightarrow w' = (w_+(0) - w_-(0))\,\delta_0 + h$ IN THE SENSE OF DISTRIBUTIONS.

OTHER POSSIBILITY:

$w(x) = g(x) + (w_+(0) - w_-(0))H(x)$ WHERE $g(x) = \begin{cases} w_-(x) & x<0 \\ w_+(x) - (w_+(0) - w_-(0)) & 0<x \end{cases}$ ... CONTINUOUS

$\Rightarrow u_w = u_g + (w_+(0) - w_-(0))\,u_H$

$\Rightarrow u_w' = u_g' + (w_+(0) - w_-(0))\,u_H' = u_h + (w_+(0) - w_-(0))\,\delta_0$

$\rightsquigarrow$ WORKS BECAUSE $\mathcal{D}'(\mathbb{R})$ IS A VECTOR SPACE (CHECK YOURSELF)

•) LET $f(x) = \ln(|x|) \in L^1_{loc}(\mathbb{R})$ $\Rightarrow$ $u_f \in \mathcal{D}'(\mathbb{R})$

$\Rightarrow$ u_f' EXISTS IN THE DISTRIBUTIONAL SENSE.

BUT $f'(x) = \frac{1}{x}$ $\Rightarrow$ NEITHER THE WEAK NOR THE CLASSICAL DERIVATIVE EXIST.

$$(u_f')(\phi) = (-1)u_f(\phi') = -\int_{\mathbb{R}} \ln|x|\,\phi'(x)\,dx = -\lim_{\varepsilon \to 0} \int_{|x|\geq\varepsilon} \ln|x|\,\phi'(x)\,dx \quad \forall \phi \in C_c^\infty(\mathbb{R})$$

$$\circ)\ -\int_{-\infty}^{-\varepsilon} \ln|x|\phi'(x)dx = -\int_{-\infty}^{-\varepsilon} \ln(-x)\phi'(x)dx = -\ln(\varepsilon)\phi(-\varepsilon) + \int_{-\infty}^{-\varepsilon} \frac{1}{x}\phi(x)dx$$

$$\circ)\ -\int_{\varepsilon}^{\infty} \ln|x|\phi'(x)dx = -\int_{\varepsilon}^{\infty} \ln(x)\phi'(x)dx = \ln(\varepsilon)\phi(\varepsilon) + \int_{\varepsilon}^{\infty} \frac{1}{x}\phi(x)dx$$

$$\Rightarrow (u_f')(\phi) = \lim_{\varepsilon\to 0} \underbrace{\ln(\varepsilon)\underbrace{(\phi(\varepsilon)-\phi(-\varepsilon))}_{=\int_{-\varepsilon}^{\varepsilon}\phi'(y)dy \leq \|\phi'\|_\infty 2\varepsilon}}_{\to 0\ (\varepsilon\to 0)} + \int_{-\infty}^{-\varepsilon} \frac{1}{x}\phi(x)dx + \int_{\varepsilon}^{\infty} \frac{1}{x}\phi(x)dx = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{1}{x}\phi(x)dx = PV[x^{-1}](\phi)$$

$$\Rightarrow u_f' = PV[x^{-1}] \quad (\text{OR } f' = PV[x^{-1}])$$

OTHER PROPERTIES:

IF $f, g \in C_c^\infty(\Omega) \Rightarrow u_{fg}(\phi) = \int_\Omega fg\phi(x)dx = \int_\Omega f(x)(g\phi)(x)dx = u_f(g\phi) \quad \forall \phi \in C_c^\infty(\Omega)$

DEF: LET $\Omega \subseteq \mathbb{R}^n$ BE A DOMAIN. FOR $g \in C^\infty(\Omega)$, $u \in \mathcal{D}'(\Omega)$, WE DEFINE $gu \in \mathcal{D}'(\Omega)$ BY

$$gu(\phi) = u(g\phi) \quad \forall \phi \in C_c^\infty(\Omega)$$

REMARK: THIS DEF CANNOT BE EXTENDED TO DEFINE PRODUCTS OF DISTRIBUTIONS.