7.2 LEBESGUE INTEGRAL:

DEF LET $E \subseteq \mathbb{R}$ (ANY SET) THE LEBESGUE MEASURE $\mathcal{L}(E)$ OF E IS DEFINED AS

$$\mathcal{L}(E) = \inf \{ \sum_{k=1}^{\infty} b_k - a_k \mid E \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \}$$

NOTE: ·) IF $E=(a,b)$, $E=[a,b]$, $E=[a,b)$ OR $E=(a,b]$ $\Rightarrow$ $\mathcal{L}(E)=b-a$ ... LENGTH OF THE INTERVAL

·) IF $E, F \subseteq \mathbb{R} \Rightarrow \mathcal{L}(E \cup F) \leq \mathcal{L}(E) + \mathcal{L}(F)$

$$\mathcal{L}(E \cup F) = \mathcal{L}(E) + \mathcal{L}(F) \text{ IF } E \cap F = \emptyset$$

·) IF $E_1, E_2, \ldots \subseteq \mathbb{R} \Rightarrow \mathcal{L}(E_1 \cup E_2 \cup \ldots) \leq \mathcal{L}(E_1) + \mathcal{L}(E_2) + \ldots$

$$\mathcal{L}(E_1 \cup E_2 \cup \ldots) = \mathcal{L}(E_1) + \mathcal{L}(E_2) + \ldots \text{ IF } E_i \cap E_j = \emptyset \ \forall i \neq j$$

·) IF E IS A COUNTABLE SET $\Rightarrow \mathcal{L}(E) = 0$

·) IF E IS THE CANTOR SET $\Rightarrow \mathcal{L}(E) = 0$

DEF: IF $f: \mathbb{R} \to \mathbb{R}$ SATISFIES A PROPERTY ON ALL OF $\mathbb{R}$ EXCEPT FOR A SET E WITH $\mathcal{L}(E)=0$ THEN WE SAY THE PROPERTY HOLDS ALMOST EVERYWHERE (A.E.)

EX: $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & \text{OTHERWISE} \end{cases}$ $\rightsquigarrow f(x) = 1$ AE

DEF A SET $E \subseteq \mathbb{R}$ IS CALLED LEBESGUE MEASURABLE IF

$$\mathcal{L}(F) = \mathcal{L}(F \cap E) + \mathcal{L}(F \setminus E) \quad \forall F \subseteq \mathbb{R}$$

NOTE ·) IF $\mathcal{L}(E)=0 \Rightarrow$ E IS $\mathcal{L}$-MEASURABLE

·) IF E IS $\mathcal{L}$-MEASURABLE $\Rightarrow E^c$ IS $\mathcal{L}$-MEASURABLE

·) OPEN AND CLOSED SETS ARE $\mathcal{L}$-MEASURABLE

·) VITALI SETS ARE NOT $\mathcal{L}$-MEASURABLE

GENERALISATION TO n-DIM:

DEF LET $E \subseteq \mathbb{R}^n$ (ANY SET). THE LEBESGUE MEASURE $\mathcal{L}(E)$ OF E IS DEFINED AS

$$\mathcal{L}(E) = \inf \{ \sum_{k=1}^{\infty} \prod_{j=1}^{n} (b_k^j - a_k^j) \mid E \subseteq \bigcup_{k=1}^{\infty} \prod_{j=1}^{n} (a_k^j, b_k^j) \}.$$

$\Rightarrow$ REMAINING DEF CARRY OVER BY REPLACING $\mathbb{R}$ BY $\mathbb{R}^n$.

DEF A FCT $f: \mathbb{R}^n \to \mathbb{R}$ IS MEASURABLE IF FOR EVERY $a \in \mathbb{R}$

$f^{-1}((-\infty, a)) = \{x \in \mathbb{R}^n \mid f(x) < a\}$ IS MEASURABLE.

EX: $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & \text{OTHERWISE} \end{cases}$ IS MEASURABLE

·) EVERY RIEMANN-INTEGRABLE FCT IS $\mathcal{L}$-MEASURABLE

·) f, g $\mathcal{L}$-MEASURABLE $\Rightarrow f g, f+g$ $\mathcal{L}$ MEASURABLE.

THE CHARACTERISTIC FCT OF A SET $A \subseteq \mathbb{R}^n$ IS DEFINED BY

$$\chi_A(\vec{x}) = \begin{cases} 1 & \vec{x} \in A \\ 0 & \text{OTHERWISE} \end{cases}$$

DEF: A FCT HAVING THE FORM $g(\vec{x}) = \sum_{i=1}^{N} c_i \chi_{A_i}(\vec{x})$ FOR SOME DISJOINT MEASUREABLE SETS $A_1, \dots, A_n \subseteq \mathbb{R}^n$ AND CONSTANTS $c_1, \dots, c_N \in \mathbb{R}$ IS CALLED A SIMPLE FCT.

IF $g(\vec{x}) \geq 0$, ITS LEBESGUE INTEGRAL IS DEFINED BY

$$\int_{\mathbb{R}^n} g(\vec{x})\, d^n\vec{x} = \sum_{i=1}^{N} c_i\, \ell(A_i)$$

IF $f: \mathbb{R}^n \to \mathbb{R}$ IS A NON-NEGATIVE MEASURABLE FCT, ITS LEBESGUES INTEGRAL IS DEFINED AS

$$\int_{\mathbb{R}^n} f(\vec{x})\, d^n\vec{x} = \sup\left\{ \int_{\mathbb{R}^n} g(\vec{x})\, d^n\vec{x} \;\middle|\; g \text{ IS SIMPLE } g \leq f \right\}.$$

EX $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{OTHERWISE} \end{cases} \rightsquigarrow \int_{\mathbb{R}^n} f(\vec{x})\, d^n\vec{x} = 0$

GENERAL: $f: \mathbb{R}^n \to \mathbb{R}$ MEASURABLE, DEFINE $f^+(\vec{x}) = \max\{f(\vec{x}), 0\}$

$$f^-(\vec{x}) = \max\{-f(\vec{x}), 0\} = -\min\{f(\vec{x}), 0\}$$

THEN $f = f^+ - f^-$ AND THE LEBESGUE INTEGRAL OF f IS GIVEN BY

$$\int_{\mathbb{R}^n} f(\vec{x})\, d^n\vec{x} = \int_{\mathbb{R}^n} f^+(\vec{x})\, d^n\vec{x} - \int_{\mathbb{R}^n} f^-(\vec{x})\, d^n\vec{x}$$

PROVIDED THAT AT LEAST ONE OF THE INTEGRALS ON THE RHS IS FINITE.

NOTE: $f = g$ A.E IF $\ell(\{\vec{x} \in \mathbb{R}^n \mid f(\vec{x}) \neq g(\vec{x})\}) = 0$

$$f = g \text{ AE.} \Leftrightarrow \int_{\mathbb{R}^n} f(\vec{x})\, d^n\vec{x} = \int_{\mathbb{R}^n} g(\vec{x})\, d^n\vec{x}$$

NOTE: CAN REPLACE THE INTEGRAL OVER $\mathbb{R}^n$ BY THE INTEGRAL OVER A MEASURABLE SET Ω.

$u: E \to \mathbb{R} \Rightarrow$ SET $\tilde{u} \equiv 0$ OUTSIDE E. AND $\tilde{u} = u$ INSIDE E

$\Rightarrow \tilde{u}: \mathbb{R}^n \to \mathbb{R}$ MEASUREABLE

$$\Rightarrow \int_{\mathbb{R}^n} \tilde{u}(\vec{x})\, d^n\vec{x} = \int_E u(\vec{x})\, d^n\vec{x}$$

1.3: L^p SPACES:

DEF: FOR $p\geq 1$, LET
$$L^p(\Omega)=\{f:\Omega\to\mathbb{R} \mid \int_\Omega |f|^p(\vec{x})\,d^n\vec{x}<\infty\}.$$

CLAIM: $L^p(\Omega)$ IS A VECTOR SPACE.

10 AXIOMS TO CHECK; SHOW 2 HERE:

- $f\in L^p(\Omega)\Rightarrow \lambda f\in L^p(\Omega)$.
$$\left(\int_\Omega |\lambda f|^p(\vec{x})\,d^n\vec{x}\right)^{1/p}=\left(|\lambda|^p\int_\Omega |f|^p(\vec{x})\,d^n\vec{x}\right)^{1/p}=|\lambda|\left(\int_\Omega |f|^p(\vec{x})\,d^n\vec{x}\right)^{1/p}<\infty$$

- $f,g\in L^p(\Omega)\Rightarrow f+g\in L^p(\Omega)$

LET $h:\mathbb{R}\to\mathbb{R}$, $x\mapsto |x|^p$ $\Rightarrow$ h IS CONVEX $\Rightarrow$ $h(\frac{1}{2}(x+y))\leq\frac{1}{2}h(x)+\frac{1}{2}h(y)$
$$\frac{1}{2^p}|x+y|^p\leq\frac{1}{2}(|x|^p+|y|^p)$$

$$\Rightarrow \left(\int_\Omega |f+g|^p(\vec{x})\,d^n\vec{x}\right)^{1/p}\leq\left(\int_\Omega 2^p(|f|^p+|g|^p)(\vec{x})\,d^n\vec{x}\right)^{1/p}$$
$$\leq 2\left(\int_\Omega |f|^p(\vec{x})\,d^n\vec{x}+\int_\Omega |g|^p(\vec{x})\,d^n\vec{x}\right)^{1/p}<\infty$$

CLAIM: $\|f\|_p=\left(\int_\Omega |f|^p(\vec{x})\,d^n\vec{x}\right)^{1/p}$ IS A NORM ON $L^p(\Omega)$.

- $\|f\|_p\geq 0\ \forall f\in L^p(\Omega)$

FOLLOWS FROM THE DEF

$\|f\|_p=0\Leftrightarrow f=0$.

ONLY TRUE IF $f\equiv 0$ AE.

$\rightsquigarrow$ NEED TO IDENTIFY EQUIVALENCE CLASSES:

f,g BELONG TO THE SAME EQUIVALENCE CLASS IF $f=g$ AE.

- $\|\lambda f\|_p=|\lambda|\,\|f\|_p\ \forall\lambda\in\mathbb{R}$

SEE $f\in L^p(\Omega)\Rightarrow\lambda f\in L^p(\Omega)$

- $\|f+g\|_p\leq\|f\|_p+\|g\|_p\ \forall f,g\in L^p(\Omega)$ ... MINKOWSKI INEQUALITY:

  - $p=1$ OBVIOUS.
  - $p>1$: YOUNG'S INEQUALITY: $ab\leq\frac{a^p}{p}+\frac{b^q}{q}$ FOR $a,b>0$ AND $\frac{1}{p}+\frac{1}{q}=1$

LET $f(u)=e^u\Rightarrow f$ CONVEX

$$\Rightarrow f(\tfrac{1}{p}u+\tfrac{1}{q}v)=f(\tfrac{1}{p}u+(1-\tfrac{1}{p})v)\leq\tfrac{1}{p}f(u)+(1-\tfrac{1}{p})f(v)=\tfrac{1}{p}f(u)+\tfrac{1}{q}f(v)$$
$$e^{\frac{1}{p}u+\frac{1}{q}v}\leq\frac{1}{p}e^u+\frac{1}{q}e^v$$

$$\Rightarrow ab=e^{\ln(a)+\ln(b)}=e^{\frac{1}{p}\ln(a^p)+\frac{1}{q}\ln(b^q)}\leq\frac{1}{p}e^{\ln(a^p)}+\frac{1}{q}e^{\ln(b^q)}=\frac{1}{p}a^p+\frac{1}{q}b^q$$

HÖLDER'S INEQUALITY: IF $f\in L^p(\Omega), g\in L^q(\Omega)$ WITH $\frac{1}{p}+\frac{1}{q}=1$ THEN

$$\int_\Omega |fg|(\vec{x})d^n\vec{x}\leq \|f\|_p\|g\|_q$$

- $f\equiv 0$ OR $g\equiv 0$: ✓
- $f\not\equiv 0$ AND $g\not\equiv 0$.

LET $\tilde{f}=\frac{f}{\|f\|_p}$, $\tilde{g}=\frac{g}{\|g\|_q}$

$$\Rightarrow \|\tilde{f}\|_p=1=\|\tilde{g}\|_q$$

$$\Rightarrow \frac{1}{\|f\|_p\|g\|_q}\int_\Omega |fg|(\vec{x})d^n\vec{x}=\int_\Omega |\tilde{f}\tilde{g}|(\vec{x})d^n\vec{x}$$

$$\leq \frac{1}{p}\int_\Omega |\tilde{f}|^p(\vec{x})d^n\vec{x}+\frac{1}{q}\int_\Omega|\tilde{g}|^q(\vec{x})d^n\vec{x}$$

$$=\frac{1}{p}+\frac{1}{q}=1$$

MINKOWSKI INEQUALITY: $\|f+g\|_p\leq\|f\|_p+\|g\|_p$

$$\|f+g\|_p^p=\int_\Omega|f+g|^p(\vec{x})d^n\vec{x}\leq\int_\Omega|f+g|^{p-1}(\vec{x})\,(|f|+|g|)(\vec{x})d^n\vec{x}$$

$$\leq \|\,|f+g|^{p-1}\|_q\,(\|f\|_p+\|g\|_p)$$

$$\leq \|f+g\|_p^{p-1}(\|f\|_p+\|g\|_p)$$

$\left[\frac{1}{p}+\frac{1}{q}=1\right]$

$\left[(p-1)q=(p-1)\frac{p}{p-1}=p\right]$

EX: $f(x)=a\chi_{[0,e]}\ (a>0)\ \Rightarrow\ \|f\|_p=\left(\int_{\mathbb{R}}|f(x)|^p dx\right)^{1/p}=(a^pe)^{1/p}=ae^{1/p}\to a\ (p\to\infty)$

DEF: FOR A MEASURABLE FCT $h:\Omega\to\mathbb{R}^n$, THE ESSENTIAL SUPREMUM IS DEFINED AS

$$\text{ess-sup}(h):=\inf\{a\in\mathbb{R}\mid \mathcal{L}(\{\vec{x}\in\Omega\mid h(\vec{x})>a\})=0\}.$$

LET $L^\infty(\Omega)=\{f:\Omega\to\mathbb{R}$ MEASURABLE $\mid$ ess-sup$(|f|)<\infty\}$ EQUIPPED WITH THE NORM $\|f\|_\infty=$ ess-sup$(|f|)$

REMARK: $L^2(\Omega)$ IS AN INNER PRODUCT SPACE

$$f,g\in L^2(\Omega)\rightsquigarrow \langle f,g\rangle=\int_{\mathbb{R}^n} fg(\vec{x})d^n\vec{x}\rightsquigarrow\|f\|_2=\sqrt{\langle f,f\rangle}$$

THM (DOMINATED CONVERGENCE)

LET $\{f_n\}$ BE A SEQUENCE IN $L^1(\Omega)$ ST
- $f_n\to f$ AE
- THERE EXISTS A NON-NEGATIVE $g\in L^1(\Omega)$ ST $|f_n|\leq g$ AE FOR ALL n.

THEN $f\in L^1(\Omega)$ AND $\int_\Omega f(\vec{x})d^n\vec{x}=\lim_{n\to\infty}\int_\Omega f_n(\vec{x})d^n\vec{x}$.