$$u_t - \Delta u = 0 \quad \text{ON } (0,\infty)\times\Omega$$
$$u(0,\vec{x}) = h(\vec{x}) \quad \forall \vec{x}\in\Omega$$

$\Big\downarrow$ u ... CLASSICAL SOLUTION

$$\int_0^\infty\int_\Omega (u_t - \Delta u)\,\varphi(t,\vec{x})\, d^n\vec{x}\, dt = 0 \quad \forall \varphi\in C_c^\infty([0,\infty)\times\Omega)$$

$\Big\downarrow$

$$\int_0^\infty\int_\Omega \left(-u\varphi_t + \langle \nabla u, \nabla\varphi\rangle\right)(t,\vec{x})\, d^n\vec{x}\, dt = \int_\Omega h(\vec{x})\,\varphi(0,\vec{x})\, d^n\vec{x} \quad \forall \varphi\in C_c^\infty([0,\infty)\times\Omega)$$

$\Big\downarrow$ $h\in L^1_{loc}(\Omega)$

$u(t,\cdot)\in H^1_0(\Omega) \quad \forall t\geq 0$

u IS A WEAK SOLUTION IF

$$\int_0^\infty\int_\Omega \left(-u\varphi_t + \langle \nabla u, \nabla\varphi\rangle\right)(t,\vec{x})\, d^n\vec{x}\, dt = \int_\Omega h(\vec{x})\,\varphi(0,\vec{x})\, d^n\vec{x} \quad \forall \varphi\in C_c^\infty([0,\infty)\times\Omega)$$

$f \in L^2(\Omega)$

$\Omega \subseteq \mathbb{R}^n$ ... BOUNDED DOMAIN

$-\Delta u = f$

POISSON EQUATION

$\downarrow$ u ... CLASSICAL SOLUTION

$$\int_\Omega (\Delta u + f)\psi(\vec{x})\, d^n\vec{x} = 0 \quad \forall \psi \in C_c^\infty(\Omega)$$

$\downarrow$

$$\int_\Omega (\langle \nabla u, \nabla \psi \rangle - f\psi)(\vec{x})\, d^n\vec{x} = 0 \quad \forall \psi \in C_c^\infty(\Omega)$$

$\downarrow$ $u \in H_0^1(\Omega)$

u IS A WEAK SOLUTION IF

$$\int_\Omega (\langle \nabla u, \nabla \psi \rangle - f\psi)(\vec{x})\, d^n\vec{x} = 0 \quad \forall \psi \in C_c^\infty(\Omega)$$

$$D_f : H_0^1(\Omega) \to \mathbb{R}$$

$$w \mapsto D_f[w] = \mathcal{E}[w] - \langle f, w \rangle_2 = \frac{1}{2}\int_\Omega \|\nabla w\|^2\, d^n\vec{x} - \int_\Omega f w(\vec{x})\, d^n\vec{x}$$

$\downarrow$ $\exists u \in H_0^1(\Omega)$ SUCH THAT $D_f[u] \le D_f[w] \quad \forall w \in H_0^1(\Omega)$

$$D_f[u] \le D_f[u + t\psi] \quad \forall \psi \in C_c^\infty(\Omega),\ t \in \mathbb{R}$$

$\downarrow$

$$\left(\frac{d}{dt} D_f[u + t\psi]\right)\Big|_{t=0} = 0 \quad \forall \psi \in C_c^\infty(\Omega)$$

$\downarrow$

$$\int_\Omega (\langle \nabla u, \nabla \psi \rangle - f\psi)(\vec{x})\, d^n\vec{x} = 0 \quad \forall \psi \in C_c^\infty(\Omega)$$

$\downarrow$

u ... WEAK SOLUTION OF $-\Delta u = f$ ON Ω