$\Omega \subseteq \mathbb{R}^n$ BOUNDED DOMAIN

$$\Downarrow \; u \in H^1_0(\Omega)$$

$$\|u\|_{H^1}^2 = \|u\|_2^2 + \int_\Omega \langle \nabla u, \nabla u \rangle(\vec{x})\, d^n\vec{x}$$
$$= \|u\|_2^2 + 2\mathcal{E}[u]$$

$$\Downarrow$$

$\exists K>0$ … DEPENDENT ON Ω
SUCH THAT
$$\|u\|_2^2 \le K^2 \mathcal{E}[u]$$

$$\Downarrow$$

$$\|u\|_{H^1}^2 \le (2+K^2)\mathcal{E}[u] \le (2+K^2)\|u\|_{H^1}^2$$

$$\Downarrow \; u \in H^1_0(\Omega),\ f \in L^2(\Omega)$$

$$\mathcal{D}_f[u] = \mathcal{E}[u] - \langle f, u \rangle_2$$

$$\Downarrow$$

$\exists! \, \bar{u} \in H^1_0(\Omega)$ ST
$$\mathcal{D}_f[\bar{u}] \le \mathcal{D}_f[w] \quad \forall w \in H^1_0(\Omega)$$

$$\Downarrow$$

$\bar{u}$ IS A WEAK SOLUTION TO
$-\Delta u = f$ IN Ω

$$\Downarrow$$

$-\Delta u = f$
HAS AT LEAST ONE WEAK SOLUTION

$$\Downarrow \; C_c^\infty(\Omega) \subseteq H^1_0(\Omega) \text{ DENSE}$$

$-\Delta u = f$
HAS EXACTLY ONE WEAK SOLUTION